Time-harmonic wave propagation with finite element and discrete exterior calculus

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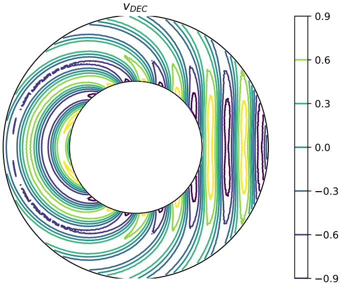

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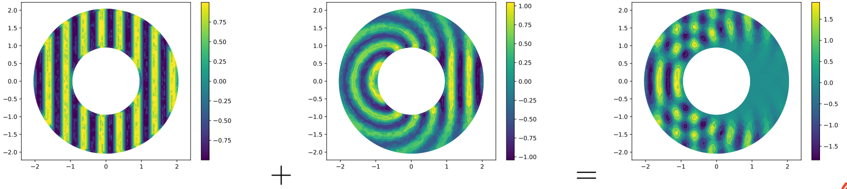
Wave propagation

- Phenomenon: propagation of time-harmonic waves.
- Analysis of time-harmonic waves in time-domain (not in frequency domain).
- As the time-domain analysis often leads to large problems, we accelerate the solution process via so-called controllability methods (Bristeau, Glowinski, and Periaux 1993, 1998).



Acoustic wave problem in mixed form

- Exterior scalar wave problem with first-order approximation of absorbing boundary:

$$
\begin{aligned}
& c^{-2}\frac{\partial v}{\partial t} - \operatorname{div}\,\mathbf{p} = 0\,, && \text{in} \quad \Omega \times (0,T)\,, && (1)\\
& \frac{\partial \mathbf{p}}{\partial t} - \operatorname{grad}\, v = 0\,, && \text{in} \quad \Omega \times (0,T)\,, && (2)\\
& v = g\,, && \text{on} \quad \gamma \times (0,T)\,, && (3)\\
& c^{-1}v + \mathbf{p}\cdot\mathbf{n} = 0\,, && \text{on} \quad \Sigma_{\text{ext}} \times (0,T)\,, && (4)\\
& v(0) = v(T)\,, \quad \mathbf{p}(0) = \mathbf{p}(T) && && (5)
\end{aligned}
$$

  where $T > 0$ is the time period, Ω is a bounded domain in $\mathbb{R}^k$, γ is the boundary on the obstacle, and Σ_{ext} is the external boundary.

- Further on, v is a scalar function, $\mathbf{p}$ is a vector function, c is a known scalar parameter (propagation velocity of the wave), g is a known source on γ and $\mathbf{n}$ is an outward unit normal vector.

FEEC vs DEC

- This presentation discusses computation of time-harmonic wave problems using a mixed formulation
- As an example, a scattering problem (in an exterior domain) is considered, and the continuous problem is first formulated in terms of differential forms.
- This presentation addresses comparison of two discretizations, the first one being based on FEEC and the second one on DEC (aka discrete differential forms, generalized finite differences).
- The discussion concentrates on scattering of a plane wave in a two-dimensional setup paying also attention on sensitivity of the numerical solution on quality of the computation grid.

Acoustic wave problem in terms of differential forms

The scalar magnitude v is associated to (dual) 0-forms and the vector magnitude $\mathbf{p}$ to (dual) 1-forms.

$$\frac{\partial v}{\partial t} - c^2 \star \mathbf{d} f = 0 \,, \qquad \text{in} \quad \Omega \times (0,T)\,, \tag{6}$$

$$(-1)^{n-1} \star \frac{\partial f}{\partial t} - \mathbf{d} v = 0 \,, \qquad \text{in} \quad \Omega \times (0,T)\,, \tag{7}$$

$$\tau_\gamma v = g \,, \qquad \text{on} \quad \gamma \times (0,T)\,, \tag{8}$$

$$\tau_\Sigma v + c \star_\Sigma \tau_\Sigma f = 0 \,, \qquad \text{on} \quad \Sigma_{\text{ext}} \times (0,T)\,, \tag{9}$$

$$v(0) = v(T)\,, \quad f(0) = f(T)\,. \tag{10}$$

The variable

$$f = \star \mathbf{p}^\flat \tag{11}$$

is now related to primal $(n-1)$-forms

Discretization, space

Assuming a triangulation Ω_d of Ω and another mesh $\tilde{\Omega}_d$ dual to it, we rewrite the equations (6-10) in a semi-discretized form for discrete differential forms that are defined as k-cochains:

$$\begin{aligned}
\frac{\partial v}{\partial t} - c^2 \mathbb{M}_n \, \mathbb{D}_{n-1} f &= 0 \, , \qquad (12)\\
(-1)^{n-1} \mathbb{M}_{n-1} \frac{\partial f}{\partial t} - \tilde{\mathbb{D}}_0 v &= 0 \, , \qquad (13)\\
\tilde{\mathbb{T}}^0_{\gamma} v &= g \, , \qquad (14)\\
\tilde{\mathbb{T}}^0_{\Sigma} v &= -c \, \mathbb{M}_{n-1,\Sigma} \, \mathbb{T}^{n-1}_{\Sigma} f \, , \qquad (15)\\
v(0) = v(T) \, , \quad f(0) &= f(T) \, . \qquad (16)
\end{aligned}$$

The unknown f is now a discrete primal $(n-1)$-form and its values are associated to $(n-1)$-faces (or edges).

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Degrees of freedom

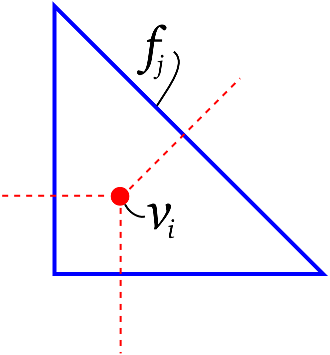

Figure: The unknown f is a primal $(n-1)$-form with its values associated to the cell faces (or edges) of the primal mesh. And the unknown v is a dual 0-form with scalar values at the vertices of the dual mesh. The vertices of the dual mesh lie at the barycenters of the primal n-cells in the case of FEEC. For DEC, the dual vertices lie at the circumcenters of the primal n-cells.

Integration by parts

$$
\begin{aligned}
(-1)^n \tilde{\mathbb{D}}_0 &= \mathbb{D}_{n-1}^{\top} - (\mathbb{T}_{\Gamma}^{n-1})^{\top} \tilde{\mathbb{T}}_{\Gamma}^{0} \\
&= \mathbb{D}_{n-1}^{\top} - (\mathbb{T}_{\Sigma}^{n-1})^{\top} \tilde{\mathbb{T}}_{\Sigma}^{0} - (\mathbb{T}_{\gamma}^{n-1})^{\top} \tilde{\mathbb{T}}_{\gamma}^{0} \,. \qquad (17)
\end{aligned}
$$

Using (17), we reorganize (12–16) to get

$$
\frac{\partial v}{\partial t} - c^2 \mathbb{M}_n \, \mathbb{D}_{n-1} f = 0 \,, \qquad (18)
$$

$$
\mathbb{M}_{n-1} \frac{\partial f}{\partial t} + \mathbb{D}_{n-1}^{\top} v + c \, (\mathbb{T}_{\Sigma}^{n-1})^{\top} \mathbb{M}_{n-1,\Sigma} \, \mathbb{T}_{\Sigma}^{n-1} f - (\mathbb{T}_{\gamma}^{n-1})^{\top} g = 0 \,, \qquad (19)
$$

$$
v(0) = v(T) \,, \quad f(0) = f(T) \,. \qquad (20)
$$

FEEC

The matrix entries in (18–20) in the case of FEEC (the vertices of the dual mesh are at the barycenters of the the primal *n*-cells):

$$\{\mathbb{M}_{n-1}\}_{ij} = \int_{T_i} \psi_i \cdot \psi_j \, \mathrm{d}x \ , \tag{21}$$

$$\{\mathbb{M}_{n}\}_{ij} = \int_{T_i} w_i w_j \, \mathrm{d}x = \frac{1}{|T_i|}\delta_{ij} \ , \tag{22}$$

$$\{\mathbb{D}_{n-1}\}_{ij} = \int_{T_i} w_i \mathrm{div}\psi_j \, \mathrm{d}x = \left\{\partial^\top\right\}_{ij} \ , \tag{23}$$

where ψ_i are the lowest-order Raviart-Thomas elements (normal components are continuous over edges) and w_i are cell-wise constant basis functions, T_i is the ith n-cell (triangle or tetrahedron) in the mesh Ω_d, $|T_i|$ is the area or volume of T_i, $\delta_{ij} = 1$, if $i = j$, and otherwise $\delta_{ij} = 0$, $\partial^\top$ is the boundary operator (or coboundary operator), where the matrix entries describe the incidence relation between oriented l-facets and $(l+1)$-facets.

DEC

And in the case of DEC, the matrix entries in (18–20) (the vertices of the dual mesh are at the circumcenters of the primal *n*-cells) are

$$\{\mathbb{M}_{n-1}\}_{ij} = \eta_{n-1,i} \frac{|* \varepsilon_i|}{|\varepsilon_i|} \delta_{ij} \ , \tag{24}$$

$$\{\mathbb{M}_{n}\}_{ij} = \eta_{n,i} \frac{|* T_i|}{|T_i|} \delta_{ij} = \eta_{n,i} \frac{1}{|T_i|} \delta_{ij} \ , \tag{25}$$

$$\{\mathbb{D}_{n-1}\}_{ij} = \left\{ \partial^\top \right\}_{ij} \ . \tag{26}$$

Above, ε_i the *i*th $(n-1)$-cell (edge or face), and $* \varepsilon_i$ is the corresponding dual 1-cell (edge), and $|\varepsilon_i|$ and $|* \varepsilon_i|$ are their volumes (areas or lengths). For the dual 0-cells, $|* T_i| = 1$. The cellwise multipliers $\eta_{k,i}$ are equal to 1 for "general" Hodge approximation.

Harmonic Hodge

The cellwise multipliers $\eta_{k,i}$ are derived using the time-harmonicity of the problem, and they have the following expressions in the cases $n = 2$, $k = n - 1, n$:

$$\eta_{1,i} = 1 - \frac{\kappa^2 \left(\frac{|* \varepsilon_i|}{2}\right)^2}{8}, \tag{27}$$

$$\eta_{2,i} = 1 + \frac{\kappa^2 r_i^2}{8}, \quad r_i^2 = \frac{|T_i|}{\pi}. \tag{28}$$

In this paper, the effective length r_i is chosen to be the radius of such a circle, the area of which is equal to the corresponding primal triangle. The parameter κ is the wave number.

Time integration

Time discretization is done in a leapfrog manner.
Discrete problem after space and time discretizations will be of the following form The forward-in-time problem will be of the following form ($\Delta t = T/N$, and $v^h = v(h\Delta t)$, $f^{h+1/2} = f((h+1/2)\Delta t)$:

$$\text{Set initial vector } (v^0, f^{\frac{1}{2}})\,. \tag{29}$$

$$\text{For } h = 0, \ldots, N-1 : \tag{30}$$

$$v^{h+1} = v^h + \mathbb{A}\, f^{h+\frac{1}{2}}\,, \tag{31}$$

$$\mathbb{H}\, f^{h+\frac{3}{2}} = \mathbb{K}\, f^{h+\frac{1}{2}} + \mathbb{B}\, v^{h+1} + \mathbb{T}\, g^{h+1}\,. \tag{32}$$

$$\text{Take } (v^N, f^{N+\frac{1}{2}})\,. \tag{33}$$

where

$$\mathbb{H} = \mathbb{M}_{n-1} + \text{boundary terms} \tag{34}$$

However, naive time stepping for such a problem leads in some cases to oscillatory solution. Instead, we will use controlled time integration using the algorithm proposed by Roland Glowinski.

Controllability

- Controllability methods by Glowinski et al. were first formulated basing on variational methods. For a scalar wave equation in a mixed formulation, controllability algorithm was proposed by Glowinski and Rossi in 2006 (variational formulation).
- Theoretical framework on the controllability techniques has further been discussed by Pauly and Rossi in 2011, generalizing the theory for a generalized Maxwell's equation in the context of differential forms.

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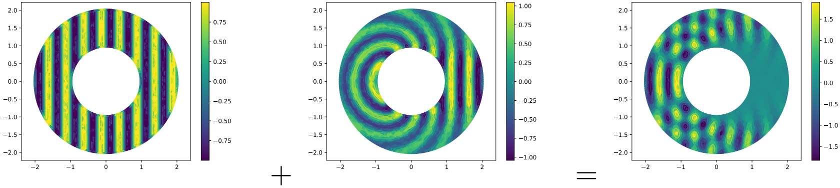
Example, Acoustic scattering

As an example, we consider a circular scatterer with ”sound-soft” boundary, and solve numerically the scattered wave as the incident wave is supposed to be given.



For the circular scatterer and in an unbounded domain, the analytical solution of the scattered plane wave can be expressed as a series solution with the help of Bessel and Hankel functions.

Meshes

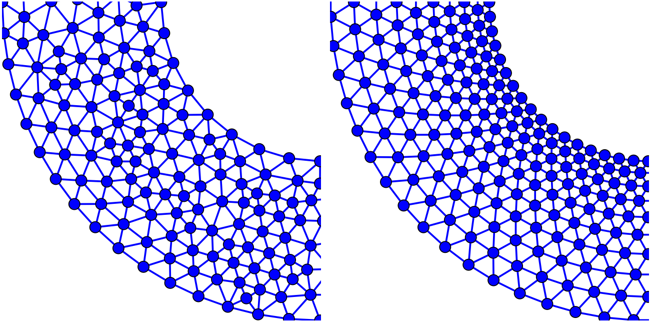
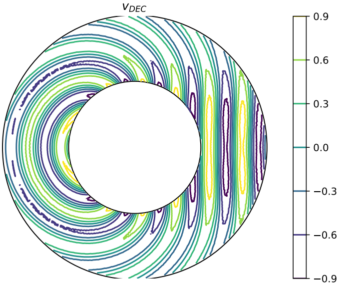


Figure: An unstructures and a structured mesh.

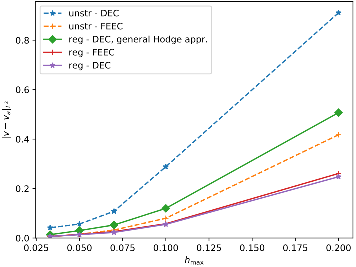
Acoustic scattering

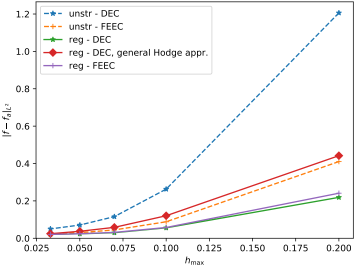


Implementation

- For implementation of numerical computations, we utilized the PyDEC software for building the simplicial complexes and the matrices (Bell and Hirani 2012).
- In the PyDEC implementation, routines for building matrices for both DEC and FEEC are given which allows us for an implementation of a general solver where the difference is only in choosing a routine for building the matrices $\mathbb{M}_k$.
- Furthermore, in the solution process, the selection of $\mathbb{M}_{n-1}$ leads to a difference in the algorithm's time stepping routine: in DEC (with a diagonal $\mathbb{M}_{n-1}$) the matrix $\mathbb{H}$ is also diagonal and thus, $\mathbb{H}^{-1}$ is fast to compute (once) and then perform a matrix multiplication with every time step. However, for FEEC, the matrix $\mathbb{H}^{-1}$ is a full matrix, and hence we solve the matrix equations utilizing a solver using LU decomposition routine `scipy.sparse.linalg.factorized`.

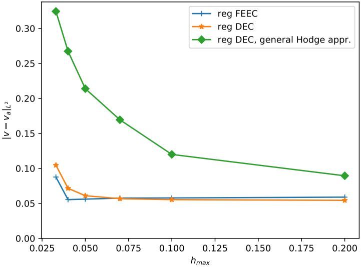
Acoustic scattering, errors

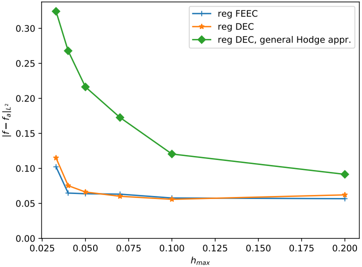




Acoustic scattering, "pollution"

- Results of the "pollution" test where $h_{\max}\omega$ was kept constant.
- Comparison of L^2 errors of v and f in the cases of the "harmonic" Hodge and the "general" Hodge approximations with respect to the maximal edge length $h_{\max}$ of mesh. If not specified in the legend, the "harmonic" Hodge approximation is applied in the DEC method. The results for the FEEC solution is plotted for comparison.





Conclusion

- For the DEC solutions, the "harmonic Hodge" approximation gave remarkably more accurate results than the general Hodge approximation.
- DEC has the advantage of fast time stepping, but
- The quality of the computational mesh affected the solution accuracy crucially in the case in discrete exterior calculus, and also slightly in the case of finite element exterior calculus.
- For the leapfrog time stepping, a CFL condition limits the size of the length of a time step.

Reference

The results of this talk have been published in
Tytti Saksa, Comparison of finite element and discrete exterior calculus in computation of time-harmonic wave propagation with controllability, Journal of Computational and Applied Mathematics, Volume 457, 2025, 116154, ISSN 0377-0427, https://doi.org/10.1016/j.cam.2024.116154.