Discrete exterior calculus on general polygonal meshes over non-flat surfaces

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Overview

We shall expose a framework presented in:

L. Ptackova and M. Outrata. Domain decomposition for mean curvature flow of surface polygonal meshes. Computer Aided Geometric Design, 127, 2026.

L. Ptackova and L. Velho. A simple and complete discrete exterior calculus on general polygonal meshes. Computer Aided Geometric Design, 88, 2021.

L. Ptackova. A discrete wedge product on general polygonal meshes. ArXiv, preprint.

The implementation can be found at:
https://github.com/lenka-ptackova/poly-DEC
https://github.com/lenka-ptackova/ddm-mcf

Motivation

Why discrete exterior calculus on general polygonal meshes?

- We are interested in geometry processing and simulations on **curved surface meshes**.
- **Exterior calculus** is a coordinate–free calculus that greatly simplifies analysis and calculations on curved spaces of differential manifolds.
- In geometric design and engineering there is a **prevalence of non–triangle surface meshes**.

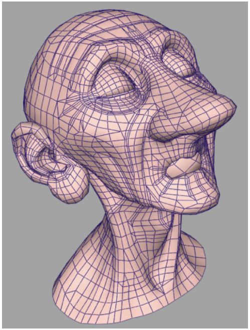
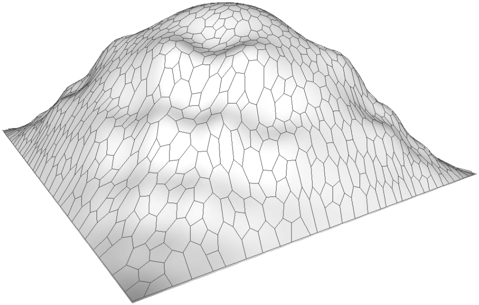


Figure: The control mesh for Geri's head used for recursive Catmull-Clark subdivision. Catmull-Clark subdivision produces meshes consisting only of quadrilaterals. Image taken from [DKT98].

A mesh and the discrete differential forms

2–dimensional orientable pseudomanifold = **mesh** → chains → cochains = **discrete differential forms**

discretization ↑



discretization ↑

2–dimensional orientable differential manifold → tangent spaces → differential forms

Our Approach

- We work directly with polygonal meshes, no combinatorial subdivision involved.
- Our construction operates solely on primal meshes, **no dual mesh** is involved.
- Our discretization is intrinsic — we only need to know the lengths of edges and the magnitudes of vector areas of faces.
- Our method is based on three basic operators: exterior derivative (coboundary operator) $\mathbf{d}$, wedge product (cup product) $\wedge$, and Hodge star operator $\star$.
- Using these three operators ($\mathbf{d}$, $\wedge$, $\star$) we derive further discrete differential operators, such as codifferential δ, Laplacian Δ, or Lie derivative.
- We evaluate the adequacy of our approach by numerical tests and by applying our operators on task such as Lie advection or Helmholtz–Hodge decomposition of vector fields.

The polygonal wedge product

- Just like the wedge product of differential forms Ω, our discrete wedge product is a metric–independent **bilinear operation** such that

$$\wedge : \Omega^k \times \Omega^l \rightarrow \Omega^{k+l}.$$

- Our polygonal wedge product satisfies the **Leibniz product rule** and is **skew–commutative**, just like its differential analog. It is actually a **proper cup product** on 2–dimensional pseudomanifolds. For a proof, see [Pta25].

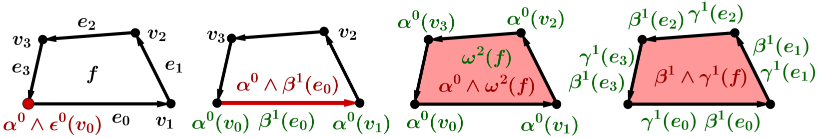


Figure: The wedge product on a quadrilateral: the product of two 0–forms is a 0–form located on vertices (far left). The product of a 0–form with a 1–form is a 1–form located on edges (center left). The product of a 0–form with a 2–form is a 2–form located on faces (center right), and the product of two 1–forms is a 2–form located on faces (far right).

The Hodge star operator

- Discrete Hodge star operator is defined as a linear operator such that

$$\star : \Omega^k \to \Omega^{2-k}.$$

- Since dual forms are atributed to primal elements, we can compute discrete wedge products of primal and dual forms and define a contraction operator later on.
- On the other hand, there is no isomorphism between the groups of k– and $(2-k)$–dimensional cells, in general. Hence our Hodge star **is not an isomorphism**, unlike the Hodge star on Riemannian manifolds.

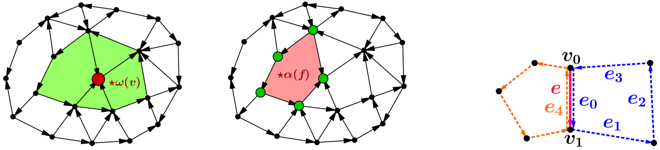


Figure: Left: Hodge dual of a 2–form ω is a 0–form $\star\omega$, whose value on vertex v (colored red) is a linear combination of values of ω on adjacent faces (colored green). Center: Hodge dual of a 0–form α is a 2–form $\star\alpha$, the value of $\star\alpha$ on face f (colored red) is a linear combination of values of α on vertices (green) of that face. Right: the Hodge dual of an 1–form on the edge e (red) is a linear combination of the given 1–form on adjacent edges (orange and blue).

The contraction operator and the Lie derivative

We define our **discrete contraction operator $\mathbf{i}_X$** on a polygonal mesh M by the following property that holds on Riemannian 2–manifolds ([Hir03, Lemma 8.2.1]):

$$\mathbf{i}_X \, \alpha = (-1)^{k(2-k)} \star (\star\alpha \wedge X^\flat), \;\; \alpha \in \Omega^k, \; k = 1, 2, \tag{1}$$

where X is a tangent vector field.

We then define **discrete Lie derivative** $\mathsf{L}_X : \Omega^k \to \Omega^k$ using the Cartan's homotopy formula and our discrete contraction operator:

$$\mathsf{L}_X \, \alpha = \mathbf{i}_X \, d\alpha + d \, \mathbf{i}_X \, \alpha, \; \alpha \in \Omega^k, \; k = 0, 1, 2, \; X \in TM. \tag{2}$$

The Leibniz product rule of the above operators with **d** is satisfied only if α or β is a closed 0–form, unfortunately.

The Codifferential and Laplacian

Just like on Riemannian n–manifolds M, the Hodge star operator is employed to define **codifferential operator** $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ by

$$\delta(\alpha^k) = (-1)^{n(k-1)+1} \star d \star \alpha.$$

Then using the codifferential operator, the **Laplace–de Rham operator** is given as

$$\Delta := \delta d + d\delta.$$

Applications: Implicit mean curvature flow

If f is a vector–valued 0–form representing the coordinates of points on a smooth surface M in R^3 and $\vec{H}$ is the mean curvature vector, then

$$\Delta f(x) = \vec{H}(f(x)), \quad x \in M \setminus \partial M.$$

Let V_0 be vertex positions of an initial mesh, we employ backward Euler method, as in [DMSB99], and solve the following linear system to find the new vertex positions with decreased mean curvature:

$$(I + dt\Delta)V_{k+1} = V_k,\ k \geq 0,\ dt > 0.$$

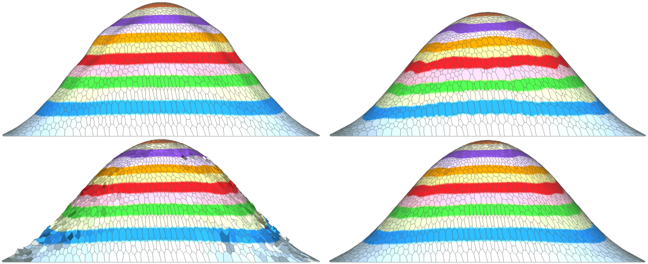


Figure: An original polygonal mesh (top left) after one iteration of curvature flow with $dt = 0.1$ using the Laplacians of [Fuj95] (top right), [AW11] (bottom left), and [PV21] (bottom right).

Applications: Implicit mean curvature flow

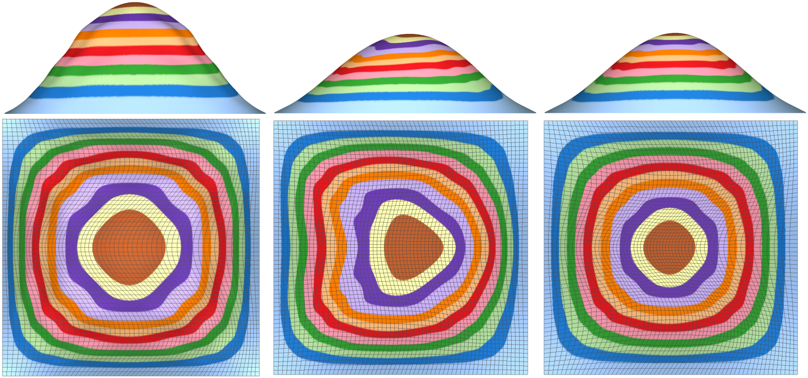


Figure: Mesh smoothing through mean curvature flow of a mesh with a texture reflecting its heightmap. In the left column is the original mesh. In the center column mesh mesh smoothed by employing the Laplacian of [Fuj95], corresponding to the standard second order five point finite difference discretization, and in the left column we employed our discrete Laplacian after 1 iteration with time step $dt = 0.05$. We can observe a significant lateral shifting of the texture when we employ the Laplacian of [Fuj95]; the shifting is caused by the non–zero tangential component of the mean curvature vector.

Applications: Discrete Helmholtz–Hodge Decomposition

The Hodge theorem states that a differential k–form ω on an oriented compact Riemannian manifold without boundary can be uniquely decomposed into three parts

$$\omega = d\alpha + \delta\beta + \gamma \tag{3}$$

for some $(k-1)$–form α, $(k+1)$–form β, and a harmonic k–form γ $(\Delta\gamma = 0)$.

If we decompose a vector field as a differential 1–form ω^1, then

1. $d\alpha$ corresponds to a curl–free component of the vector field,
2. $\delta\beta$ corresponds to a divergence–free (incompressible) component,
3. γ corresponds to a harmonic component.

This is the **three component form** of the HHD. As mentioned in [Bhatia et al. 2013], some applications employ the **two component form** of the HHD, where the harmonic component is "included" either into the curl–free or the divergence–free part.

Applications: Discrete Helmholtz–Hodge Decomposition

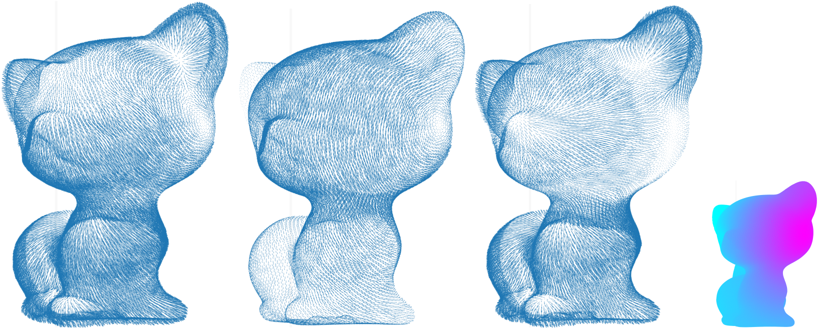


Figure: Helmholtz–Hodge decomposition on a hexagonal mesh. From left to right: original vector field X, its incompressible vorticial part, its curl-free part, and corresponding vector potential β in pseudo-colors. The color magenta correctly reflects the distance to the center of CCW rotation of the vector field.

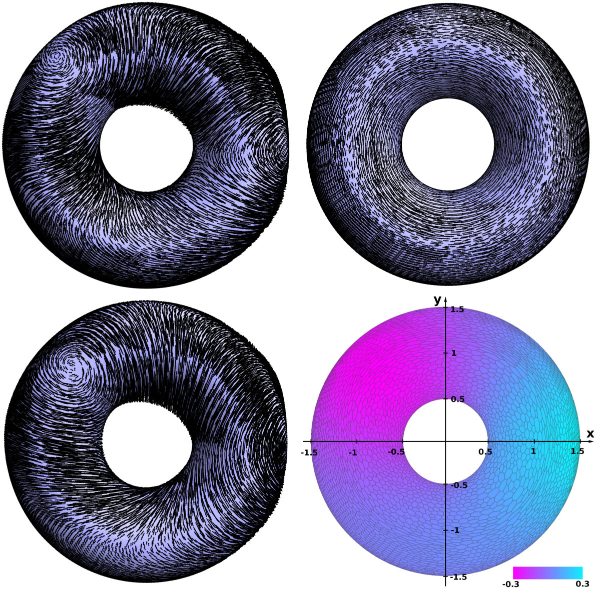


Figure: HHD of an incompressible vector field X (top left) on a torus. The original vector field is $X = X_H + X_R$, where $X_H = (-y, x, 0)$ is a harmonic field, and X_R is a rotational vector field. The calculated decomposition consists of harmonic part $\gamma^\sharp$ (top right) and rotational part $(\delta\beta)^\sharp$ (bottom left).

Applications: Lie Advection

We can model advection of a conserved q–form β in a flow generated by vector field X by solving the Lie advection equation:

$$\frac{\partial \beta}{\partial t} + \mathsf{L}_X \, \beta = 0.$$

Thus to advect a discrete q–form β by the flow of a vector field X, we can iterate over discrete solutions using a simple forward Euler method:

$$\beta_{k+1} = \beta_k - dt \, \mathsf{L}_X \, \beta_k, \ k = 0, \ldots, \tag{4}$$

where dt is the time step, k is the number of iterations.

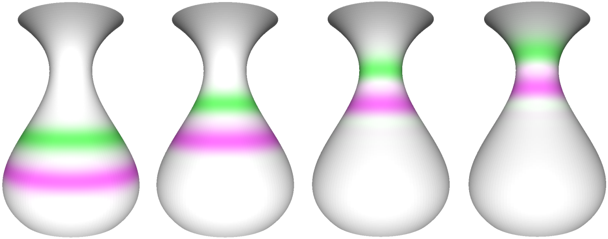


Figure: Lie advection of a color function (a (R,G,B)–valued 0–form) on a mesh of a vase.

Applications: Lie Advection

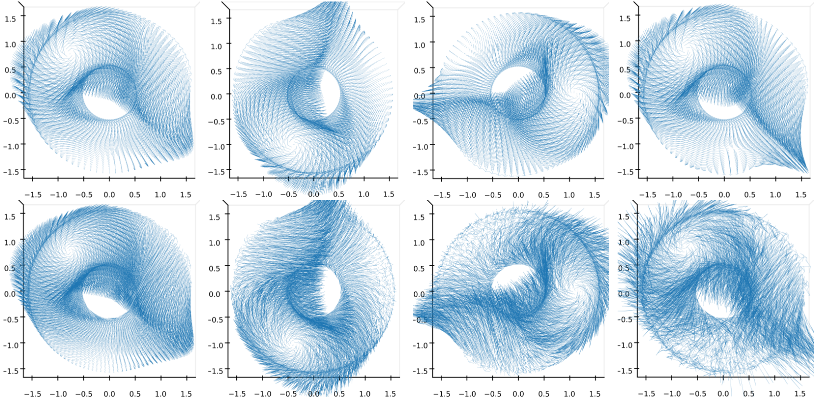


Figure: Lie advection of a vorticial vector field Y along the flow of $X = (-y, y, 0)^\top$, which is a Killing vector field with periodic orbit. We show the advection with $dt = 0.01$ after 0, 200, 400, 635 iterations on a regular quad mesh with 5k vertices (top row) and jittered quad mesh with 8k vertices (bottom row). We observe that on regular meshes the advection is almost flawless, whereas on irregular meshes some artifacts develop and not even mesh refinement nor decreasing the time steps solves the issue completely.

Numerical behavior

- Experiments show at least linear convergence of our wedge product and Hodge star operator both in L^2 and L^∞ norms, but they are exact on constant differential forms on planar surfaces.
- Experimental convergence of Lie derivative was observed on regular meshes for 0–, 1–, and 2–forms (area potentials). On irregular meshes the errors stay rather constant for 1– and 2–forms, and decrease at least linearly on 0–forms.
- Our codifferential and Laplace–de Rham operator exhibit convergent behavior on regular grids. On irregular meshes the error of approximation stays rather constant after an initial decrease. However, our Laplacian is **linearly precise**, i.e., it is zero on linear forms in the plane (unlike the standard second order five point finite difference discretization).

Ongoing and Future Work

- **Discretization of vector fields and their procerssing** — investigate the pros and cons of discretizing vector fields as real-valued 1-forms or as vector-valued 1-forms.
- **2D fluid flow simulation** — apply the framework for fluid flow simulation, study proper boundary conditions.
- **DEC on 3–dimensional pseudomanifolds** — examine the possibility of extending our framework to 3–dimensional to volumetric meshes made of tetrahedrons or 3–dimensional (topological) cubes or more general polyhedrons.

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