Finite element form-valued forms
– Towards an extended periodic table –

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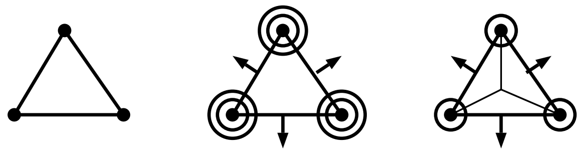
March 26, 2026

AknowledgementRoyal Society University Research Fellowship (URF\R1\221398)

ERC starting grant (101164551): **GeoFEM** (Geometric Finite Element Methods)

Finite elements: from scalar fields (50s, 60s)

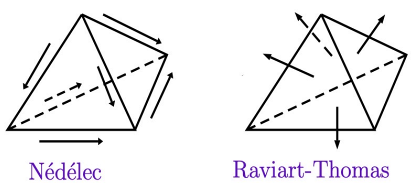


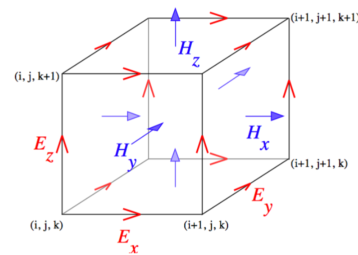
C^0 C^1 C^1

> Degrees of Freedom (DoFs): *discrete patterns*, where quantities are discretized
> shape functions: extend DoFs to piecewise functions inside

DoFs uniquely determine shape functions

to vector fields (70s-80s)

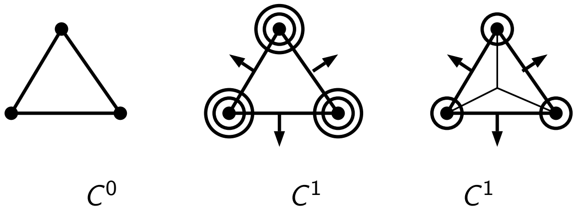




Yee scheme: electric field on edges, magnetic fields on faces

(Fig: https://ammar-hakim.org)

Finite elements: from scalar fields (50s, 60s)



Degrees of Freedom (DoFs): *discrete patterns*, where quantities are discretized
shape functions: extend DoFs to piecewise functions inside

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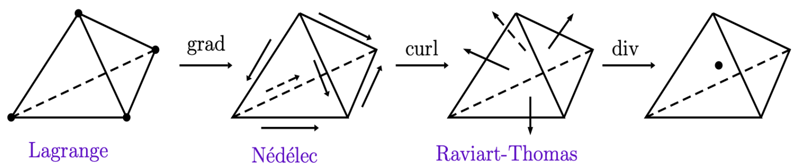
to differential forms (Bossavit 80s; Hiptmair 90s; Arnold, Falk, Winther 2000s...) k-forms on k-cells



grad **Lag** $\subset$ **Ned**, curl **Ned** $\subset$ **RT**, div **RT** $\subset$ **DG**, and exactness / cohomology

Differential forms

On $\mathbb{R}^n$, a k-form is an element in $C^\infty(\Omega) \otimes \mathrm{Alt}^k$.

- A general k-form: $\omega = \sum_I f_I \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.
- antisymmetric: $dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad dx_i \wedge dx_i = 0$.

$$f = f \Rightarrow \text{0-form}, \ \mathbb{R}$$
$$\omega = udx + vdy + wdz \Rightarrow \text{1-form}, \ \mathbb{V}$$
$$\eta = ady \wedge dz + bdz \wedge dx + cdx \wedge dy \Rightarrow \text{2-form}, \ \mathbb{V}$$
$$\mu = fdx \wedge dy \wedge dz \Rightarrow \text{3-form}, \ \mathbb{R}$$

- Differential operator: $d\omega = \sum_I \left(\sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

$$d = \mathrm{grad}(\text{0-form}), \mathrm{curl}(\text{1-form}), \mathrm{div}(\text{2-form}), 0(\text{3-form})$$

- Traces $\iota_F^* \omega : \sum_I (f_I|_F) \ (dx_{i_1}|_F) \ \wedge \ \cdots \wedge \ (dx_{i_k}|_F)$.

$$\iota^* = \mathrm{value}(\text{0-form}), \omega \cdot t(\text{1-form}), \omega \cdot n(\text{2-form}), 0(\text{3-form})$$

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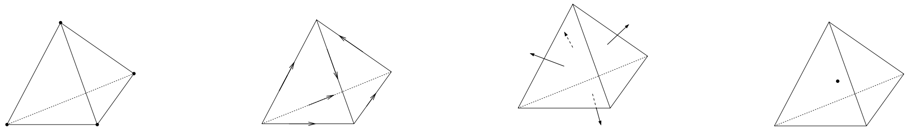
de Rham complex

$$C^\infty \Lambda^k := C^\infty \otimes \mathrm{Alt}^k$$

$$0 \longrightarrow C^\infty \Lambda^0 \xrightarrow{d^0} C^\infty \Lambda^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^\infty \Lambda^n \longrightarrow 0.$$

$$0 \longrightarrow C^\infty \xrightarrow{\text{grad}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{curl}} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{div}} C^\infty \longrightarrow 0.$$

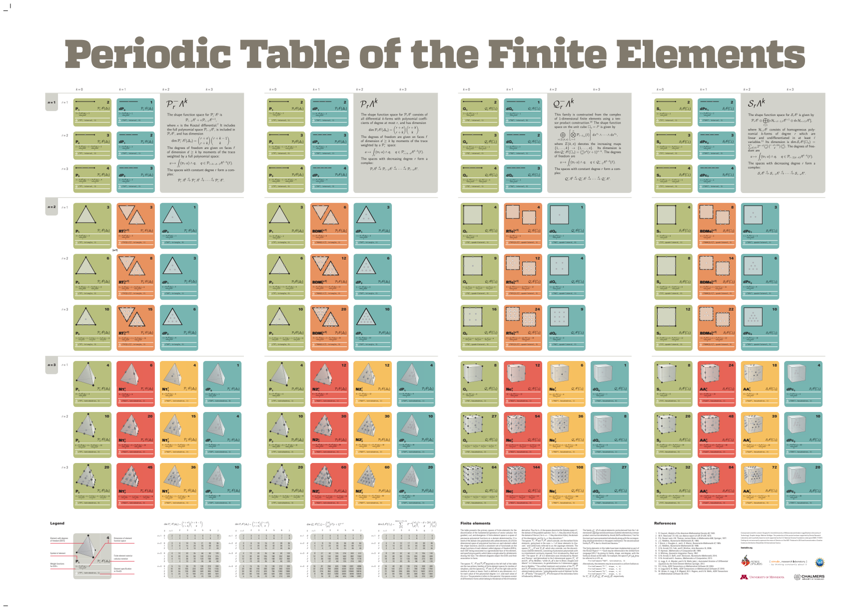
Whitney Forms



0-form	1-form	2-form	3-form	k-form
Lagrange	**Nédélec**	**Raviart-Thomas**	**DG**	**Whitney forms**
$\mathcal{P}_1 = \mathbb{R} + \boldsymbol{x}\cdot\mathbb{V}$	$\mathbb{V} + \boldsymbol{x}\times\mathbb{V}$	$\mathbb{V} + \boldsymbol{x}\otimes\mathbb{R}$	$\mathbb{R}$	$\mathrm{Alt}^k + \kappa\mathrm{Alt}^{k+1}$
dim $= 4$	dim $= 6$	dim $= 4$	dim $= 1$	dim $= \binom{n+1}{k+1}$
DoFs: $u \mapsto u(\boldsymbol{x})$	DoFs: $\boldsymbol{v} \mapsto (\boldsymbol{v}\cdot\boldsymbol{t})\vert_e$	DoFs: $\boldsymbol{w} \mapsto (\boldsymbol{w}\cdot\boldsymbol{n})\vert_f$	DoFs: $\boldsymbol{w} \mapsto \int w$	DoFs: $\omega \mapsto \iota_\sigma^*\omega$
value continuity	t continuity	n continuity	discontinuous	tr $:= \iota^*$ continuity

Koszul construction: $\kappa(dx_{i_1}\wedge\cdots\wedge dx_{i_k}) \mapsto \sum_s (-1)^{s-1} x_{i_s}(dx_{i_1}\wedge\cdots \cancel{dx_{i_s}}\cdots\wedge\cdots dx_{i_k})$

Finite element periodic table: any dimension, any k-forms, any polynomial degrees



Arnold, Logg 2014, SIAM news

Reference for Finite Element Exterior Calculus (FEEC): Arnold, Falk, Winther (2006) Acta Numerica; Arnold, Falk, Winther (2010) Bulletin of AMS; Arnold (2018) SIAM book

Finite Element Tensor Calculus (FETC)

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

Discrete analogues of such tensors with symmetries and differential structures?

Finite Element Tensor Calculus (FETC)

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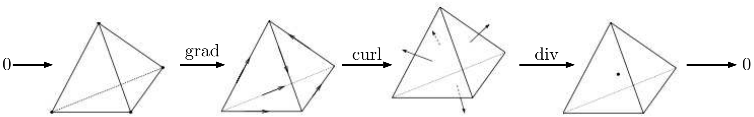
Discrete analogues of such tensors with symmetries and differential structures?

A special case: differential forms (fully skew-symmetric tensors), exterior derivatives



Raviart-Thomas (1977), Nédélec (1980) in numerical analysis
Bossavit (1988): differential forms and complex
Hiptmair (1999), Arnold, Falk, Winther (2006): systematic study, "Finite Element Exterior Calculus"

Finite Element Tensor Calculus (FETC)

stress, strain tensors, dislocation density, disclination density in continuum mechanics,
metric, curvature (scalar, Ricci, Weyl, Riemann, Cotton...), torsion in differential geometry etc.

Discrete analogues of such tensors with symmetries and differential structures?

Example: metric

$g_{ij} = g_{ji}$.

Example: Riemannian tensor

$R_{ab;cd} = -R_{ba;cd} = -R_{ab;dc}$, $\quad R_{ab;cd} = R_{cd;ab}$, $\quad R_{ab;cd} + R_{bc;ad} + R_{ca;bd} = 0$.
(algebraic Bianchi identity)

Quiz: in 4D ($a, b, c, d = 1, 2, 3, 4$), number of independent components of g_{ab} and $R_{ab;cd}$?

Tensors fit in complexes

Example: elasticity (Kröner, Calabi) complex

displacement formulation

Kröner's continuum description of dislocations/defects, internal stress

$$0 \longrightarrow C^\infty \otimes \mathbb{V} \xrightarrow{\text{def}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0$$

displacement *strain (metric)* *stress (curvature)* *force*

intrinsic elasticity (Ciarlet et al.)

Hellinger-Reissner principle of elasticity

$\mathbb{V} := \mathbb{R}^3$ vectors, $\mathbb{S} := \mathbb{R}^{3\times 3}_{\text{sym}}$ symmetric matrices

$$\operatorname{def} u := 1/2(\nabla u + \nabla u^T), \quad (\operatorname{def} u)_{ij} = 1/2(\partial_i u_j + \partial_j u_i).$$

$$\operatorname{inc} g := \nabla \times g \times \nabla, \quad (\operatorname{inc} g)^{ij} = \epsilon^{ikl}\epsilon^{jst}\partial_k\partial_s g_{lt}.$$

$$\operatorname{div} v := \nabla \cdot v, \quad (\operatorname{div} v)_i = \partial^j u_{ij}.$$

g metric $\Rightarrow$ $\operatorname{inc} g$ linearized Einstein tensor ($\backsimeq$ Riem $\backsimeq$ Ric in 3D)

$\operatorname{inc} \circ \operatorname{def} = 0$: Saint-Venant compatibility
$\operatorname{div} \circ \operatorname{inc} = 0$: Bianchi identity

Form-value Forms / Double Forms

On $\mathbb{R}^n$, a (k,ℓ)-form is an element in $\underbrace{(C^\infty(\Omega)\otimes \mathrm{Alt}^k)}_{k \text{ forms}} \underbrace{\otimes \mathrm{Alt}^\ell}_{\ell\text{-form valued}} =: C^\infty(\Omega)\otimes \mathrm{Alt}^{k,\ell}$.

$$g = g_{ij} dx^i \otimes dx^j \quad \text{(sym (1,1)-form)}, \quad R = R_{ijpq} dx^i \wedge dx^j \otimes dx^p \wedge dx^q \quad \text{(sym (2,2)-form)}.$$

In three dimensions:

	dx	dy	dz
dx	*	*	*
dy	*	*	*
dz	*	*	*

(1,1) forms

	dx	dy	dz
$dy \wedge dz$	*	*	*
$dz \wedge dx$	*	*	*
$dx \wedge dy$	*	*	*

(2,1) forms

	$dy \wedge dz$	$dz \wedge dx$	$dx \wedge dy$
dx	*	*	*
dy	*	*	*
dz	*	*	*

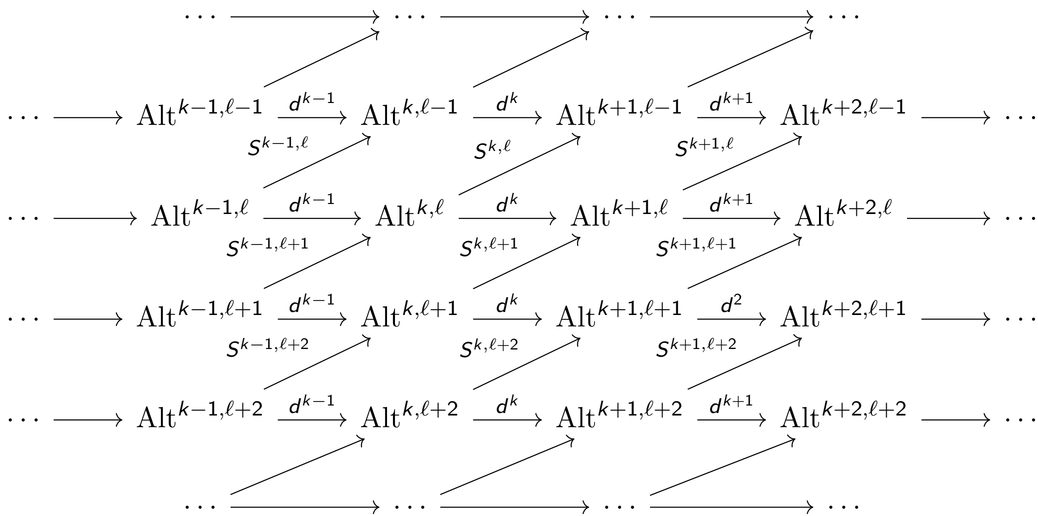
(1,2) forms

	$dy \wedge dz$	$dz \wedge dx$	$dx \wedge dy$
$dy \wedge dz$	*	*	*
$dz \wedge dx$	*	*	*
$dx \wedge dy$	*	*	*

(2,2) forms

- For Whitney (k,ℓ)-forms, we introduce $\mathcal{P}^-\mathrm{Alt}^{k,\ell} = \underbrace{(\mathrm{Alt}^k + \kappa \mathrm{Alt}^{k+1})}_{\text{Whitney } k\text{-forms}} \otimes \underbrace{\mathrm{Alt}^\ell}_{\text{Const } \ell\text{-forms}}$

$\mathrm{Alt}^{k,\ell} := \mathrm{Alt}^k \otimes \mathrm{Alt}^\ell$ ℓ-form-valued k-forms; (k, ℓ)-forms)



$\mathcal{S}^{k,\ell} : \mathrm{Alt}^{k,\ell} \to \mathrm{Alt}^{k+1,\ell-1}$

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \otimes (dx^{j_1} \wedge \cdots \wedge dx^{j_\ell})$$

$$\mapsto \sum_s (-1)^s (dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_s}) \otimes dx^{j_1} \wedge \cdots \cancel{dx^{j_s}} \cdots \wedge dx^{j_\ell}$$

$\mathcal{S}_\dagger^{k,\ell} : \mathrm{Alt}^{k,\ell} \to \mathrm{Alt}^{k-1,\ell+1}$

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \otimes (dx^{j_1} \wedge \cdots \wedge dx^{j_\ell})$$

$$\mapsto \sum_s (-1)^s (dx^{i_1} \wedge \cdots \cancel{dx^{i_s}} \cdots \wedge dx^{i_k} \wedge) \otimes dx^{j_1} \wedge \cdots \wedge dx^{j_\ell} \wedge dx^{i_s}$$

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$$
\begin{array}{ccccccccccc}
 & & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & & \\
 & & & \nearrow & & \nearrow & & \nearrow & & & \\
\cdots & \longrightarrow & \mathrm{Alt}^{k-1,\ell-1} & \xrightarrow{d^{k-1}} & \mathrm{Alt}^{k,\ell-1} & \xrightarrow{d^{k}} & \mathrm{Alt}^{k+1,\ell-1} & \xrightarrow{d^{k+1}} & \mathrm{Alt}^{k+2,\ell-1} & \longrightarrow & \cdots \\
 & & & \nearrow S^{k-1,\ell} & & \nearrow S^{k,\ell} & & \nearrow S^{k+1,\ell} & & & \\
\cdots & \longrightarrow & \mathrm{Alt}^{k-1,\ell} & \xrightarrow{d^{k-1}} & \mathrm{Alt}^{k,\ell} & \xrightarrow{d^{k}} & \mathrm{Alt}^{k+1,\ell} & \xrightarrow{d^{k+1}} & \mathrm{Alt}^{k+2,\ell} & \longrightarrow & \cdots \\
 & & & \nearrow S^{k-1,\ell+1} & & \nearrow S^{k,\ell+1} & & \nearrow S^{k+1,\ell+1} & & & \\
\cdots & \longrightarrow & \mathrm{Alt}^{k-1,\ell+1} & \xrightarrow{d^{k-1}} & \mathrm{Alt}^{k,\ell+1} & \xrightarrow{d^{k}} & \mathrm{Alt}^{k+1,\ell+1} & \xrightarrow{d^{2}} & \mathrm{Alt}^{k+2,\ell+1} & \longrightarrow & \cdots \\
 & & & \nearrow S^{k-1,\ell+2} & & \nearrow S^{k,\ell+2} & & \nearrow S^{k+1,\ell+2} & & & \\
\cdots & \longrightarrow & \mathrm{Alt}^{k-1,\ell+2} & \xrightarrow{d^{k-1}} & \mathrm{Alt}^{k,\ell+2} & \xrightarrow{d^{k}} & \mathrm{Alt}^{k+1,\ell+2} & \xrightarrow{d^{k+1}} & \mathrm{Alt}^{k+2,\ell+2} & \longrightarrow & \cdots \\
 & & & \nearrow & & \nearrow & & \nearrow & & & \\
 & & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & &
\end{array}
$$

Tensor symmetries encoded in diagrams

Example: Riemannian tensor

$\ker(S^{2,2}) \subset \mathrm{Alt}^{2,2}$: symmetry of Riemannian tensor (algebraic Bianchi identity)
$R_{ab;cd} = -R_{ba;cd} = -R_{ab;dc}$, $R_{ab;cd} = R_{cd;ab}$, $R_{ab;cd} + R_{bc;ad} + R_{ca;bd} = 0.$

$$
\dim(\ker(S^{2,2})) = \dim \mathrm{Alt}^{2,2} - \dim \mathrm{Alt}^{3,1} = \begin{cases} 1 & \text{in 2D} \quad \text{Gauss curvature}, \\ 6 & \text{in 3D} \quad \text{Ricci/Einstein}, \\ 20 & \text{in 4D} \quad \text{Riemann} \\ \cdots & \end{cases}
$$

Differential structures of tensor fields

Derive complexes of tensors from de Rham complexes. Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction.

(B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006, Arnold,KH 2021, Čap,KH 2023)

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Alt}^{0,J-1} & \xrightarrow{d} & \mathrm{Alt}^{1,J-1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathrm{Alt}^{n,J-1} & \longrightarrow & 0 \\
 & & & \nearrow_{S^{0,J}} & & \nearrow_{S^{1,J}} & & \nearrow_{S^{n-1,J}} & & & \\
0 & \longrightarrow & \mathrm{Alt}^{0,J} & \xrightarrow{d} & \mathrm{Alt}^{1,J} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \mathrm{Alt}^{n,J} & \longrightarrow & 0
\end{array}
$$

$\mathcal{S}^{k-1,k} : \Lambda^{k-1,k} \to \Lambda^{k,k-1}$: bijective. Left $\mathcal{S}$ operators: injective. Right $\mathcal{S}$ operators: surjective.

Output:

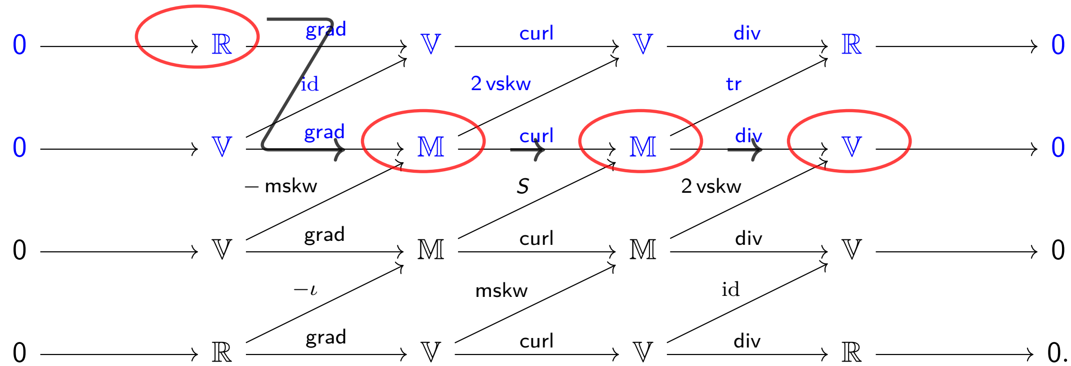
$$
0 \longrightarrow \ker(S_{\dagger}^{-1,J}) \longrightarrow \cdots \longrightarrow \ker(S_{\dagger}^{J-2,J}) \xrightarrow{d} \quad \xrightarrow{S^{-1}} \quad \xrightarrow{d} \ker(S^{J,J}) \longrightarrow \cdots \longrightarrow \ker(S^{n,J}) \longrightarrow 0.
$$

BGG reduction: $\ker(S_{\dagger}^{k,\ell}) \text{ "=" } \mathrm{Alt}^{k,\ell} - \mathcal{S}\mathrm{Alt}^{k-1,\ell+1}$

$S_{\dagger}$: adjoint of S

3D EXAMPLES

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix



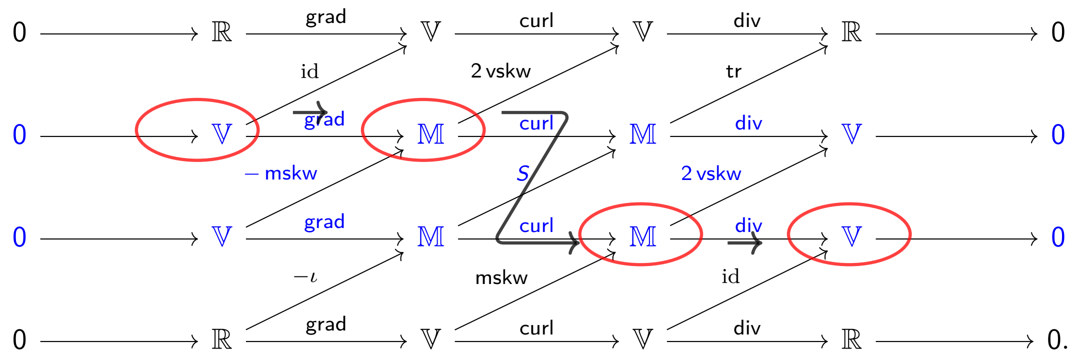
Hessian complex:

$$0 \longrightarrow C^\infty \xrightarrow{\text{hess}} C^\infty(\mathbb{S}) \xrightarrow{\text{curl}} C^\infty(\mathbb{T}) \xrightarrow{\text{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

biharmonic equations, plate theory, Einstein-Bianchi system of general relativity

3D examples

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix



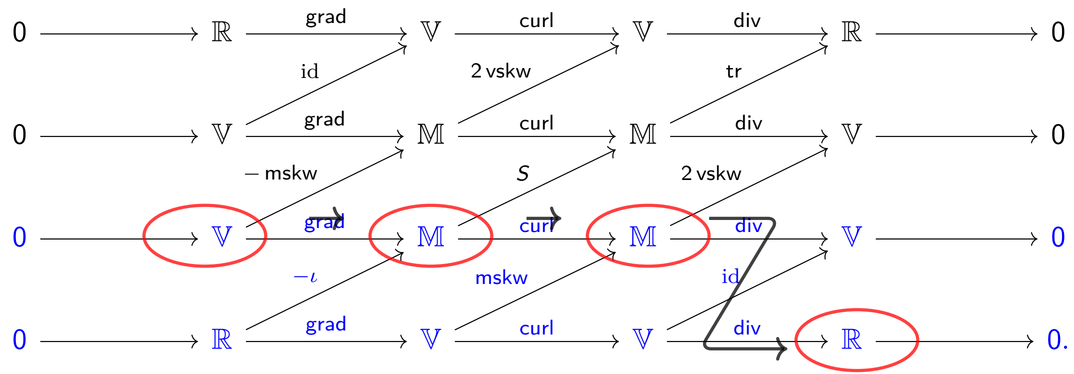
elasticity complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\text{def}} C^\infty(\mathbb{S}) \xrightarrow{\text{inc}} C^\infty(\mathbb{S}) \xrightarrow{\text{div}} C^\infty(\mathbb{V}) \longrightarrow 0.$$

elasticity, defects, metric, curvature

3D examples

$\mathbb{R}$: scalar $\mathbb{V}$: vector $\mathbb{M}$: matrix $\mathbb{S}$: symmetric matrix $\mathbb{T}$: trace-free matrix



divdiv complex:

$$0 \longrightarrow C^\infty(\mathbb{V}) \xrightarrow{\text{dev grad}} C^\infty(\mathbb{T}) \xrightarrow{\text{sym curl}} C^\infty(\mathbb{S}) \xrightarrow{\text{div div}} C^\infty \longrightarrow 0.$$

plate theory, elasticity

Discretization of complexes: finite elements and splines

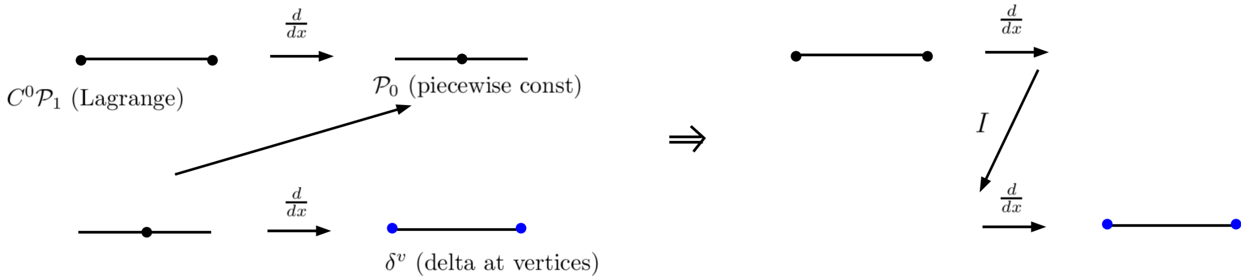
Existing results (selected, most conforming):

- 2D: stress, strain — Arnold–Winther 2002; Chen–Hu–Huang 2014; Christiansen–KH 2018; ...
- 3D: elasticity, Hessian, divdiv — Arnold–Awanou–Winther 2008; Chen–Huang 2020–21; Hu–Liang–Lin 2023; ...
- Higher dimensions: nD, (k, ℓ)-forms, tensor product constructions — Bonizzoni–KH–Kanschat–Sap 2023; ...
- Other directions: conformal complexes — KH–Lin–Shi 2023

Main question: Canonical finite elements — analogue of Whitney forms?

Goal: correct cohomology, discrete topological / geometric structures...

1D finite elements BGG



Generalizing finite elements: back to de Rham's currents

Key idea: For higher dimensions, introducing currents , in addition to functions

Currents = measures (Dirac delta, distributions)
Encode geometry in a *codimensional way*

Intuition: A point cloud ∼ 3-form, curve cloud ∼ 2-form, surface cloud ∼ 1-form

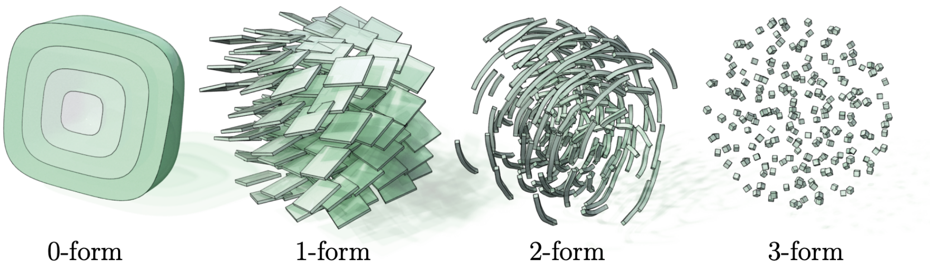
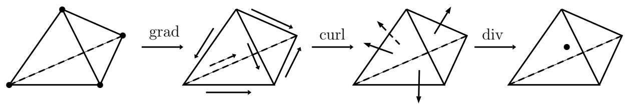


Figure 2.4 Differential k-forms can be represented by clouds of codimension-k geometries.

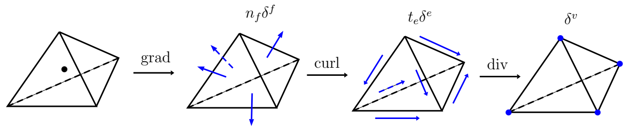
Figure: *Exterior Calculus in Graphics*, Wang–Nabizadeh–Chern, SIGGRAPH 2023

This suggests a natural extension of finite elements beyond functions.

Perspectives of currents (distributions)



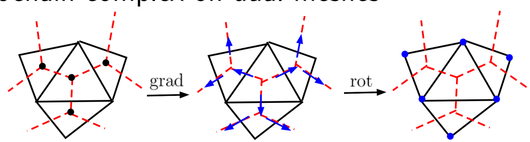
standard finite elements



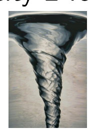
Braess, Schöberl 2008

Perspectives:

- Topological: k-th space $\cong$ $(n-k)$-chains, $d^k \cong \partial_{n-k}$
- Finite element: dual point of view, complex of degrees of freedom
- Discrete exterior calculus: cochain complex on dual meshes



- Fluid: point vortex, vortex lines (vorticity 2-form: delta on codimension 2)

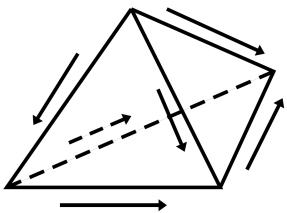



- Applications: equilibrated residual error estimators (Braess–Schöberl 2008)
- Analysis: Licht 2017 (double complex)

Gathering pieces to solve the puzzle

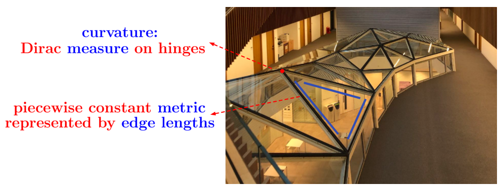
As the development of FEEC, several individual ingredients are already in the literature.

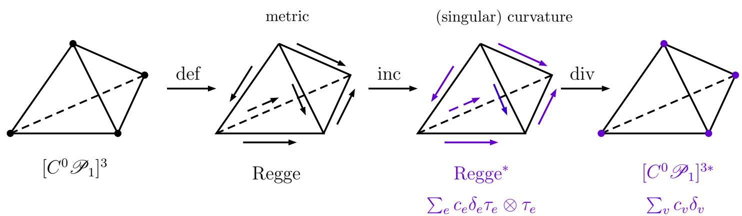
Christiansen 2011: Regge calculus = finite elements



- Regge element. discrete metric
- $\dim \mathbb{S} = 6$
- DoFs: $A_{tt}|_e = \boldsymbol{t}_e \cdot A \cdot \boldsymbol{t}_e$
- $\#\text{DoFs} = 6.$
- $t - t$ continuity

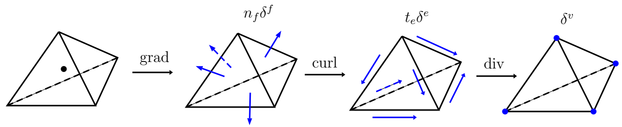
Regge finite element v.s. Regge Calculus (*'General relativity without coordinates'*)





Schöberl and collaborators: **distributional finite element schemes**

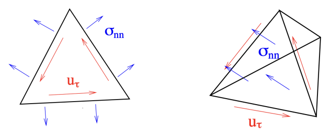
- Equilibrated residual error estimator



distributional finite elements, Braess-Schöberl 2008, Licht 2017

- Tangential Displacement Normal-Normal Stress (TDNNS) method for elasticity

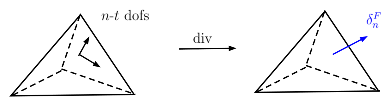
$$\sigma = -\varepsilon(\boldsymbol{u}), \quad \nabla\cdot\sigma = \boldsymbol{f}. \quad \sigma: \text{ symmetric matrix}$$



div σ: tangential distribution   Schöberl-Sinwel 2007

- Mass-Conserving mixed Stress (MCS) method for Stokes equations

$$\sigma = -\nabla\boldsymbol{u}, \quad \nabla\cdot\sigma + \nabla p = \boldsymbol{f}. \quad \sigma: \text{ trace-free matrix}$$



div σ: normal distribution   Gopalakrishnan-Lederer-Schöberl 2020

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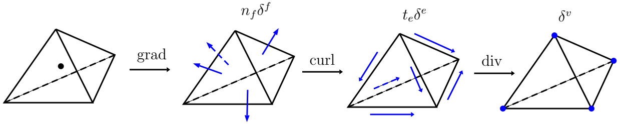

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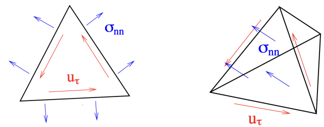

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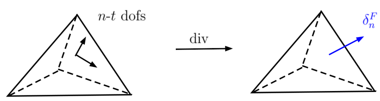


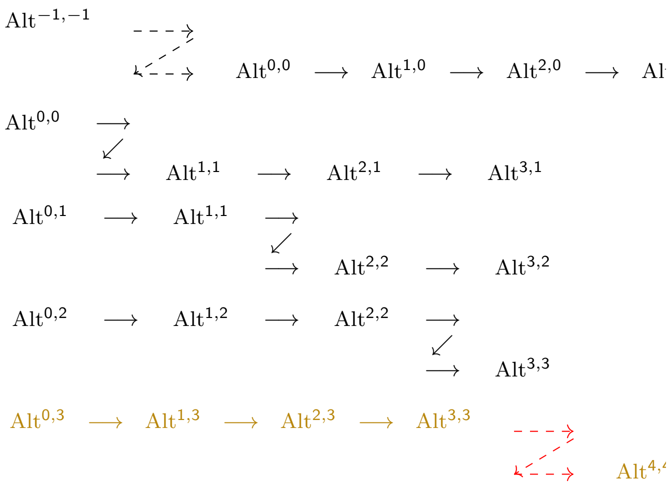
div σ: normal distribution Gopalakrishnan-Lederer-Schöberl 2020

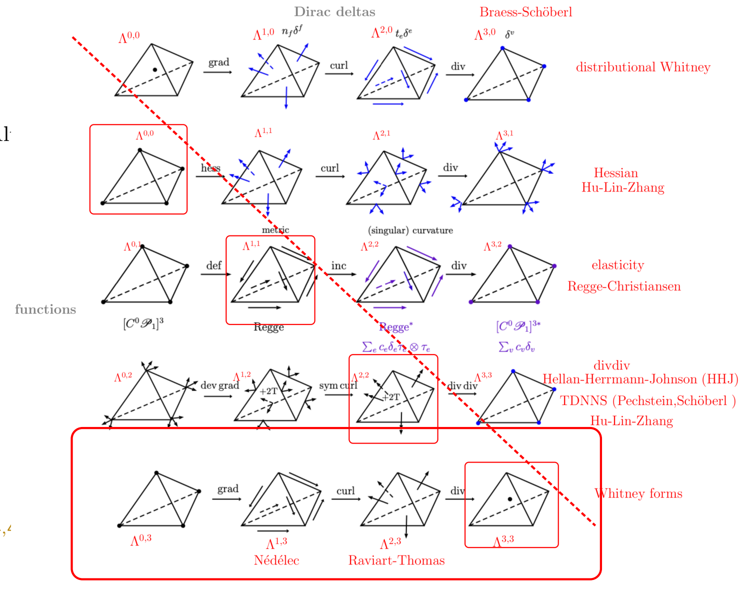
This work

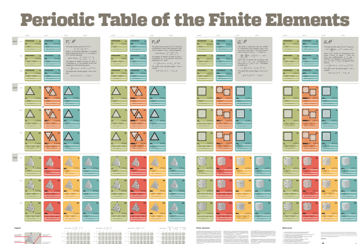
Unify the above constructions in the same picture, as *form-valued forms*?

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms: Construction.* arXiv: 2503.03243 (2025)







classical Finite Element Exterior Calculus

Nédélec, Raviart–Thomas, Whitney, Bossavit, Hiptmair, Arnold, Falk, Winther...

Towards a finite element periodic table for tensors

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$\mathrm{Alt}^{-1,-1}$

$\mathrm{Alt}^{0,0} \longrightarrow \mathrm{Alt}^{1,0} \longrightarrow \mathrm{Alt}^{2,0} \longrightarrow \mathrm{Al}$

$\mathrm{Alt}^{0,0} \longrightarrow$

$\longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \mathrm{Alt}^{2,1} \longrightarrow \mathrm{Alt}^{3,1}$

$\mathrm{Alt}^{0,1} \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow$

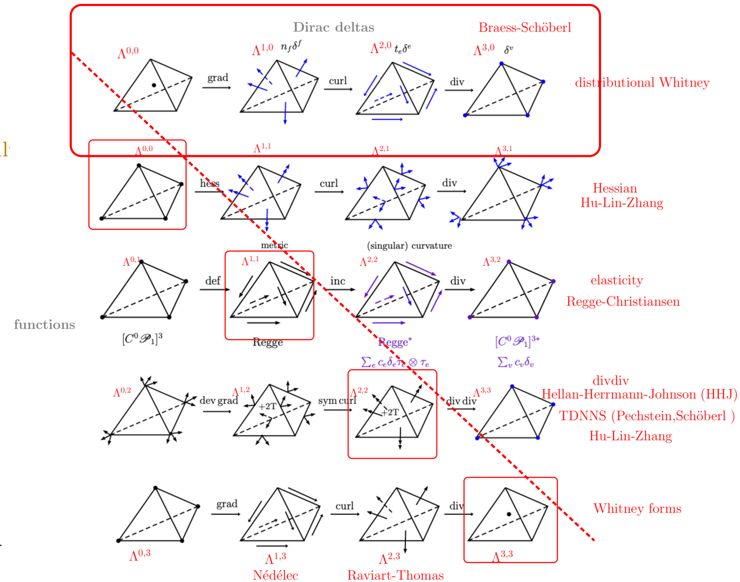
$\longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \mathrm{Alt}^{3,2}$

$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow$

$\longrightarrow \mathrm{Alt}^{3,3}$

$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3}$

$\mathrm{Alt}^{4,4}$



distributional de Rham complex (currents).

Braess, Schöberl 2008: equilibrated residual error estimator

Licht 2017: double complex

Dietrich Braess

Joachim Schöberl

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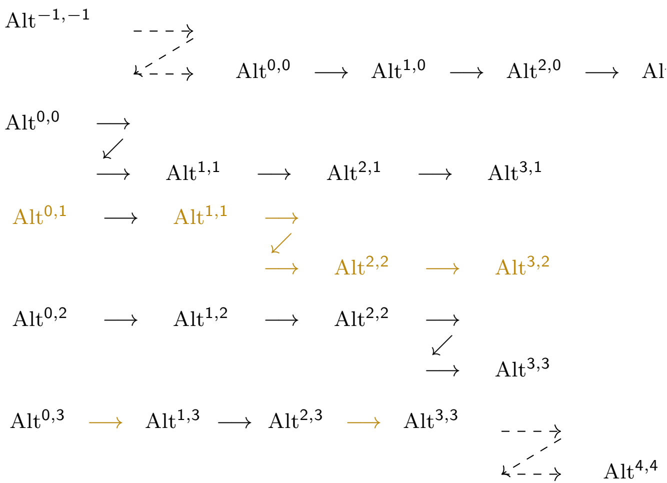
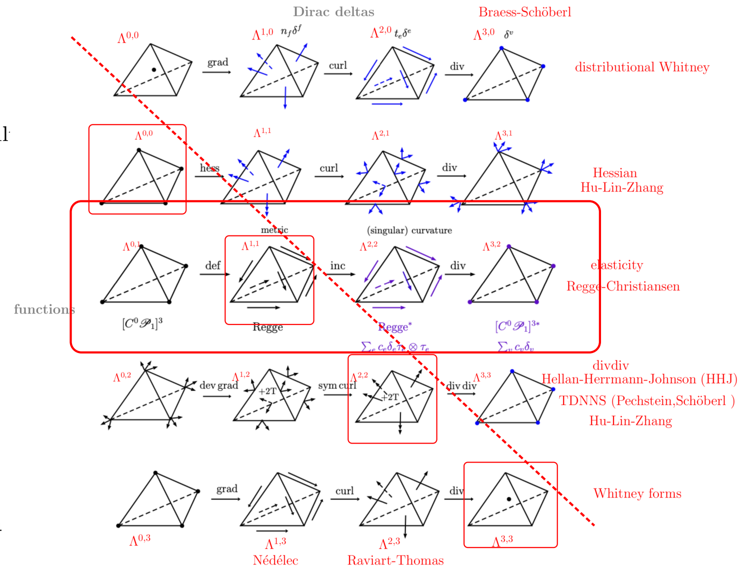




Christiansen's interpretation of Regge calculus as finite elements

Regge calculus (quantum & numerical gravity) :
edge length as metric, angle deficit as curvature

Regge finite element : piecewise constant symmetric tensor field

Tullio Regge

Snorre Christiansen

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms: Construction.* arXiv: 2503.03243 (2025)

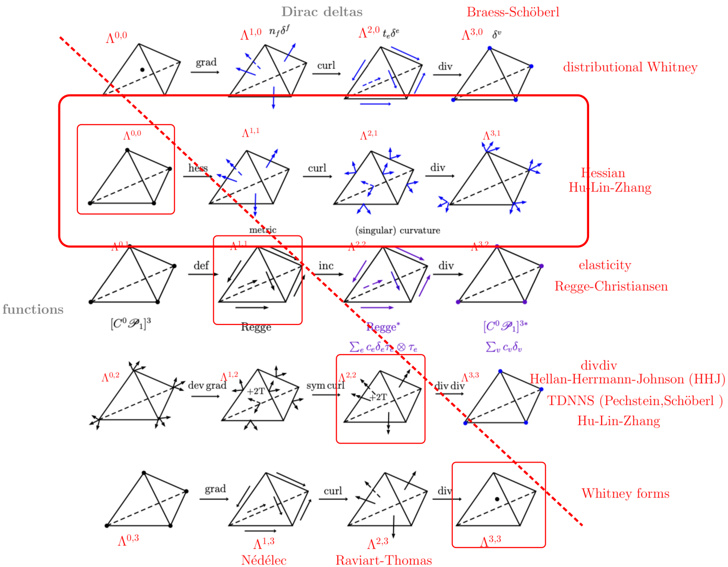
$$\mathrm{Alt}^{-1,-1} \dashrightarrow \mathrm{Alt}^{0,0} \longrightarrow \mathrm{Alt}^{1,0} \longrightarrow \mathrm{Alt}^{2,0} \longrightarrow \cdots$$

$$\mathrm{Alt}^{0,0} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \mathrm{Alt}^{2,1} \longrightarrow \mathrm{Alt}^{3,1}$$

$$\mathrm{Alt}^{0,1} \longrightarrow \mathrm{Alt}^{1,1} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \mathrm{Alt}^{3,2}$$

$$\mathrm{Alt}^{0,2} \longrightarrow \mathrm{Alt}^{1,2} \longrightarrow \mathrm{Alt}^{2,2} \longrightarrow \swarrow \longrightarrow \mathrm{Alt}^{3,3}$$

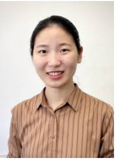
$$\mathrm{Alt}^{0,3} \longrightarrow \mathrm{Alt}^{1,3} \longrightarrow \mathrm{Alt}^{2,3} \longrightarrow \mathrm{Alt}^{3,3} \dashrightarrow \mathrm{Alt}^{4,4}$$



Hessian complex, unified structures identified.

Kaibo Hu, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).

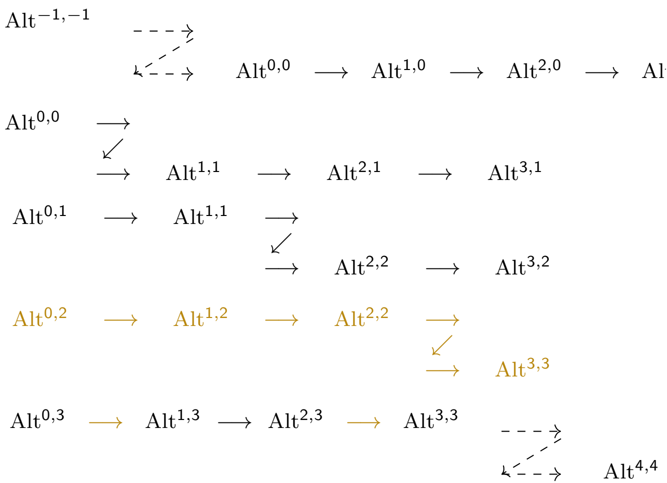
Ting Lin

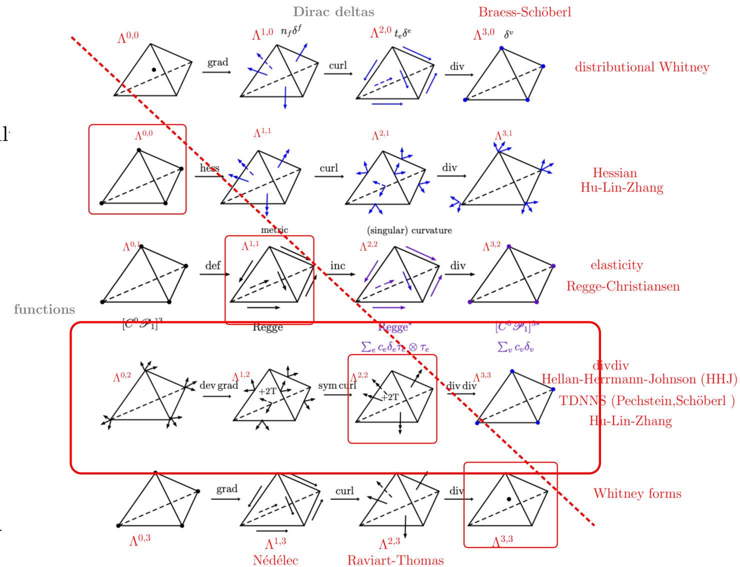


Qian Zhang

Towards a finite element periodic table for tensors

KH, Ting Lin. *Finite element form-valued forms: Construction.* arXiv: 2503.03243 (2025)





divdiv complex , dual to Hessian complex.
TDNNS for elasticity (Schöberl, Sinwel 2007), Hellan-Herrmann-Johnson (HHJ) element for plate.

Astrid Pechstein Joachim Schöberl

KH, Ting Lin, Qian Zhang. *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM Journal on Applied Algebra and Geometry (2025).

Towards a finite element periodic table for tensors

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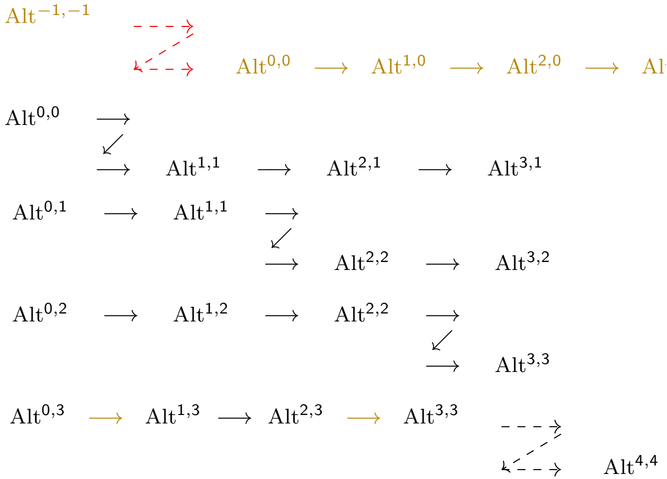
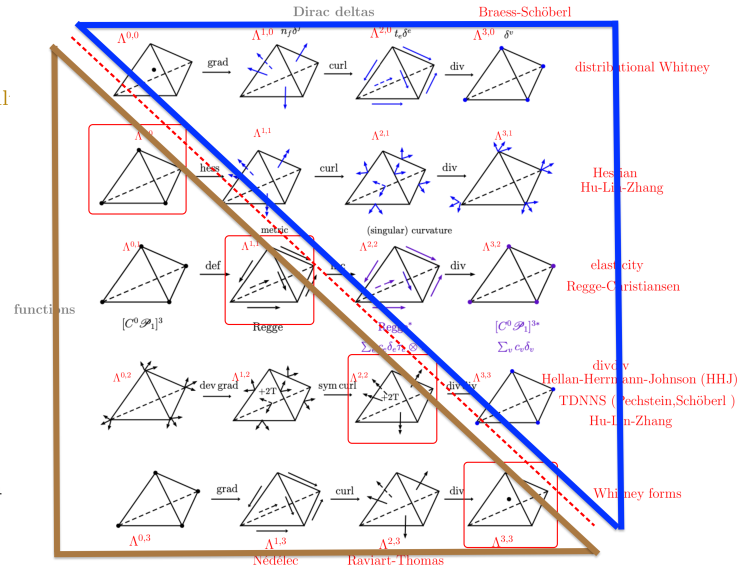




Patterns, Symmetries, Duality.

functions (classical finite elements) v.s. Dirac measures (currents). any dimension, any degree.

Georges de Rham

Classical Finite Element Periodic Table (last row) is the special case of the generalised Table where all spaces are piecewise polynomials.

Summary & Outlook

Takeaway: Finite elements = discrete geometry

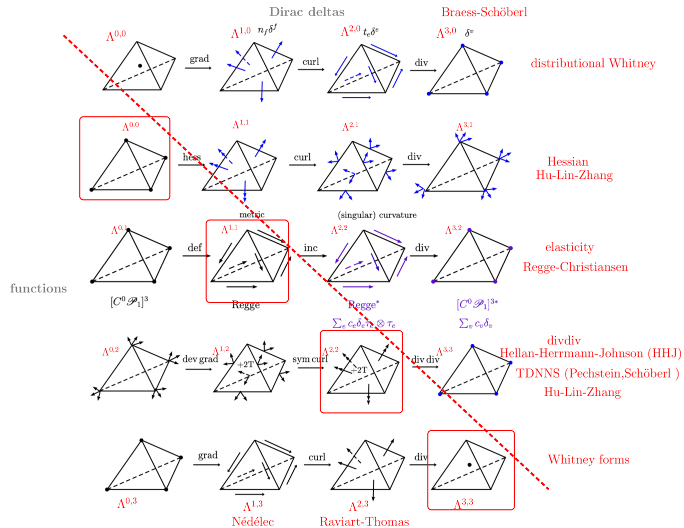
Forms → tensors → unified periodic table

What we built:

- Finite element form-valued forms (k, ℓ)
- Arbitrary symmetries via $\mathcal{S}$, $\mathcal{S}_\dagger$
- Arbitrary polynomial degree

Why it matters:

- Invariant under pullbacks (Piola)
  natural for manifolds, surfaces, shells
- Discretizes full BGG diagrams
  curvature, torsion, Bianchi identities
- Robust w.r.t. geometry/physics
  Schöberl: model reduction by discretization



Outlook:

- Cohomology theory (beyond low order, higher dimensions)
- Conformal complexes ($\mathbb{S} \cap \mathbb{T}$, Weyl tensors)

References

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  framework & continuous construction
- Čap, Hu (2023). *BGG sequences with weak regularity and applications*, Found. Comput. Math.
  general framework, conformal complexes
- Hu, Lin, Zhang (2025). *Distributional Hessian and divdiv complexes on triangulation and cohomology*. SIAM J. Appl. Algebra Geom.
  distributional BGG complexes in 2D/3D, cohomology results
- **Hu, Lin (2025).** ***Finite element form-valued forms*****. arXiv:2503.03243**
  systematic construction
- Berchenko-Kogan, Gawlik (2025). *Finite element spaces of double forms*. arXiv:2505.17243
  different perspectives for $\iota^*\iota^*$-conforming elements
- Gopalakrishnan, Hu, Schöberl (2025). *A 2-complex containing Sobolev spaces of matrix fields*. arXiv:2507.11869
  2-complexes