

MODELLING GENERALIZED CONTINUA VIA HOMOLOGICAL ALGEBRA

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OUTLINE

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2	Complexes from complexes: continuum models	11
3	Double complexes: mixed dimensional geometry	23

DE RHAM COMPLEXES: ELECTROMAGNETISM

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DE RHAM COMPLEX (3D VERSION)

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$

$$d^0 := \text{grad}, \quad d^1 := \text{curl}, \quad d^2 := \text{div}.$$

- ▶ complex property: $d^k \circ d^{k-1} = 0$, $\Rightarrow \mathcal{R}(d^{k-1}) \subset \ker(d^k)$,
 $\text{curl} \circ \text{grad} = 0 \Rightarrow \mathcal{R}(\text{grad}) \subset \ker(\text{curl})$, $\text{div} \circ \text{curl} = 0 \Rightarrow \mathcal{R}(\text{curl}) \subset \ker(\text{div})$
- ▶ cohomology: $\mathcal{H}^k := \ker(d^k) / \mathcal{R}(d^{k-1})$,
 $\mathcal{H}^0 := \ker(\text{grad})$, $\mathcal{H}^1 := \ker(\text{curl}) / \mathcal{R}(\text{grad})$, $\mathcal{H}^2 := \ker(\text{div}) / \mathcal{R}(\text{curl})$
- ▶ exactness (contractible domains): $\ker(d^k) = \mathcal{R}(d^{k-1})$, i.e., $d^k u = 0 \Rightarrow u = d^{k-1} v$
 $\text{curl } u = 0 \Rightarrow u = \text{grad } \phi$, $\text{div } v = 0 \Rightarrow v = \text{curl } \psi$.

In higher dimensions,

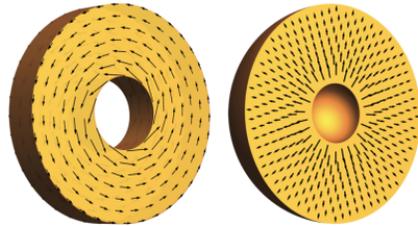
$$\dots \longrightarrow \Lambda^{k-1} \xrightarrow{d^{k-1}} \Lambda^k \xrightarrow{d^k} \Lambda^{k+1} \longrightarrow \dots$$

Λ^k : differential k -forms, d^k : exterior derivatives

DE RHAM COMPLEX AND TOPOLOGY

Key fact:

$\dim \mathcal{H}^k \sim$ number of holes (Betti number)



Examples:

- ▶ $\dim \mathcal{H}^1 = 1$, divergence-free field that is not a curl
- ▶ $\dim \mathcal{H}^2 = 1$, curl-free field that is not a gradient

(figure from *Finite Element Exterior Calculus*, Arnold, SIAM 2008)

FROM COMPLEXES TO PDES

Formal adjoint of operators:

$$\text{grad}^* = -\text{div}, \quad \text{curl}^* = \text{curl}, \quad \text{div}^* = -\text{grad}.$$

$$\int_{\Omega} \text{grad } u \cdot v = - \int_{\Omega} u \text{div } v + \text{bound. term}, \quad \int_{\Omega} \text{curl } u \cdot v = \int_{\Omega} u \cdot \text{curl } v + \text{bound. term}$$

$$(\text{grad } u, v) = (u, -\text{div } v), \quad (\text{curl } u, v) = (u, \text{curl } v)$$

Formal adjoint of de Rham complex:

$$0 \longleftarrow C^{\infty}(\Omega) \xleftarrow{-\text{div}} C^{\infty}(\Omega; \mathbb{R}^3) \xleftarrow{\text{curl}} C^{\infty}(\Omega; \mathbb{R}^3) \xleftarrow{-\text{grad}} C^{\infty}(\Omega) \longleftarrow 0.$$

$$d_2^* := -\text{div}, \quad d_1^* := \text{curl}, \quad d_0^* := -\text{grad}.$$

FROM COMPLEXES TO PDES

Examples of PDEs from complexes: Hodge-Laplacian problems.

$$(d^{k-1}d_{k-1}^* + d_k^*d^k)u = f.$$

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$$0 \rightleftarrows C^\infty(\Omega) \begin{array}{c} \xrightarrow{\text{grad}} \\ \xleftarrow{-\text{div}} \end{array} C^\infty(\Omega; \mathbb{R}^3) \quad C^\infty(\Omega; \mathbb{R}^3) \quad C^\infty(\Omega) \quad 0.$$

Hodge-Laplacian problem:

$$-\text{div grad } u = f.$$

Poisson equation.

Variational form (energy):

$$\inf_u \frac{1}{2} \|\nabla u\|^2 - \int_\Omega fu.$$

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Hodge-Laplacian problem:

$$-\text{grad div } v + \text{curl curl } v = f.$$

Maxwell equations.

Variational form (energy):

$$\inf_v \frac{1}{2} (\|\text{curl } v\|^2 + \|\text{div } v\|^2) - \int_\Omega f v.$$

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$$\text{curl curl } v - \text{grad div } v = f.$$

Maxwell equations.

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Hodge-Laplacian problem:

$$-\text{div grad } u = f.$$

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Variational form (energy):

$$\inf_u \frac{1}{2} \|\nabla u\|^2 - \int_\Omega fu.$$

Why are complexes useful for PDEs?

- ▶ well-posedness: analytic properties + standard variational argument

Stokes problem:

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx - \int_{\Omega} p \nabla \cdot \mathbf{v} \, dx &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, & \forall \mathbf{v}, \\ \int_{\Omega} \nabla \cdot \mathbf{u} \, q \, dx &= 0, & \forall q. \end{aligned}$$

Well-posedness via

Ladyzhenskaya-Babuška-Brezzi inf-sup condition:

$$\inf_{q \in L^2/\mathbb{R}} \sup_{\mathbf{v} \in \mathbf{H}_0^1} \frac{\int \operatorname{div} \mathbf{v} \, q \, dx}{\|\mathbf{v}\|_{H^1} \|q\|_{L^2}} \geq \gamma > 0$$

From exact de Rham complex:

$$0 \longrightarrow V^0 \xrightarrow{\operatorname{grad}} V^1 \xrightarrow{\operatorname{curl}} V^2 \xrightarrow{\operatorname{div}} V^3 \longrightarrow 0$$

velocity pressure

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velocity
pressure

- ▶ discretization: structure-preservation!

Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area.

– James Clerk Maxwell, 1873

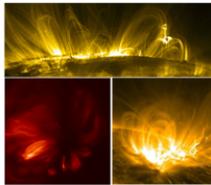
Discrete Differential Forms (Bossavit 1988, Hiptmair 1999, ...),
 Finite Element Exterior Calculus (Arnold, Falk, Winther 2006, ...)

discrete spaces fit into de Rham complexes.

MOTIVATION: STRUCTURE-PRESERVING DISCRETISATION

Fundamental question in plasma physics: given initial data, what does the system evolve to?

heating of solar corona, plasma equilibria (magnetic configurations) etc.



Magneto-friction (simplified MHD) :

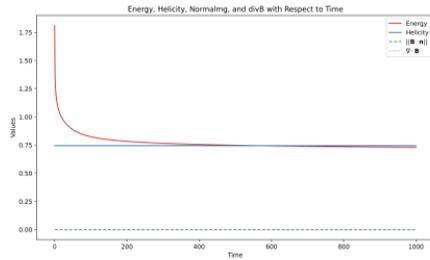
$$\begin{aligned}\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) &= 0, \\ \mathbf{j} &= \nabla \times \mathbf{B}, \\ \mathbf{u} &= \tau \mathbf{j} \times \mathbf{B}.\end{aligned}$$

Energy decay

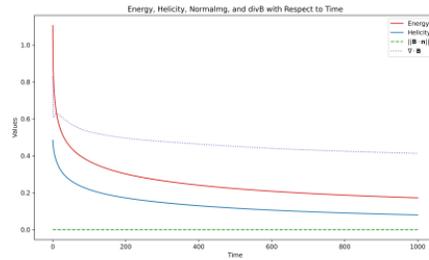
$$\frac{1}{2} \frac{d}{dt} \|\mathbf{B}\|^2 = -\tau \|\mathbf{B} \times \mathbf{j}\|^2.$$

Helicity conservation

$$\frac{d}{dt} \mathcal{H}_m = 0, \quad \text{with } \mathcal{H}_m := \int \mathbf{A} \cdot \mathbf{B} \, dx, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$



Helicity-preserving scheme



CG scheme (non-preserving)

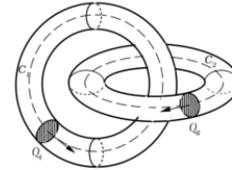
Initial data: Hopf fibration

- *Topology-preserving discretization for the magneto-frictional equations arising in the Parker conjecture*, M. He, P. E. Farrell, KH, B. Andrews, SISC (2025).

A TOPOLOGICAL MECHANISM

Magnetic helicity: Helmholtz, Kelvin, Woltjer, Moffatt...
for any potential \mathbf{A} satisfying $\nabla \times \mathbf{A} = \mathbf{B}$,

$$\mathcal{H}_m := \int_{\Omega} \mathbf{A} \cdot \mathbf{B} \, dx$$



Arnold inequality (V.I. Arnold 1974): helicity provides lower bound for energy

$$\left| \int \mathbf{A} \cdot \mathbf{B} \, dx \right| \leq C \int |\mathbf{B}|^2 \, dx$$

Proof. Cauchy-Schwarz $|\int \mathbf{A} \cdot \mathbf{B} \, dx| \leq \|\mathbf{A}\|_{L^2} \|\mathbf{B}\|_{L^2} + \text{Poincaré inequality } \|\mathbf{A}\|_{L^2} \leq C \|\nabla \times \mathbf{A}\|_{L^2}.$



Vladimir I. Arnold

Knots: topological barriers preventing energy decay.
Mechanism lost if algorithms do not preserve helicity.

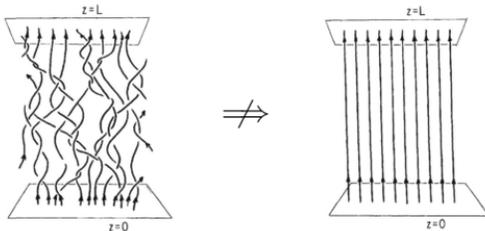
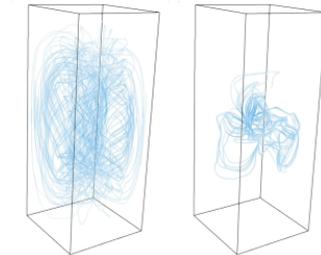


Fig. Pontin, Hornig, Living Rev. Sol. Phys. 2020.

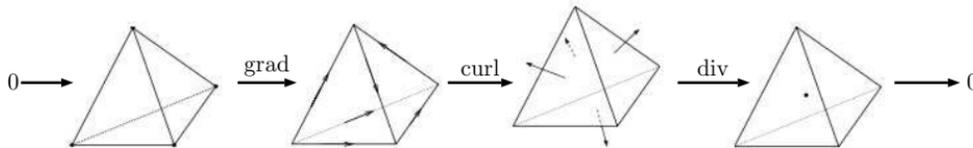


HOW TO PRESERVE HELICITY: DISCRETE DE RHAM COMPLEX

- ▶ Raviart–Thomas (1977), Nédélec (1980): Early finite elements
- ▶ Bossavit (1988): Differential forms and complex
- ▶ Hiptmair (1999), Arnold, Falk, Winther (2006): Systematic Finite Element Exterior Calculus

Classical Whitney forms

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \longrightarrow 0.$$



$\mathbf{E}_h, \mathbf{H}_h$

\mathbf{B}_h

- ▶ Faraday's law $\partial_t \mathbf{B}_h + \nabla \times \mathbf{E}_h = 0$ holds exactly $\implies \frac{d}{dt}(\nabla \cdot \mathbf{B}_h) = 0$.
- ▶ Introducing projection $\mathbf{H}_h = Q_{L^2} \mathbf{B}_h \implies (\mathbf{u}_h \times \mathbf{H}_h, Q_{L^2} \mathbf{B}_h) = 0$.

RELIABLE NUMERICAL COMPUTATION

Differential complex perspective

Modelling using *differential complexes*, discretizing entire complexes, more than individual fields

Example: discretizing Maxwell's equations by discretizing de Rham complex

Structure-aware modelling: Extend this principle to elasticity (BGG) and coupling (Čech)

- ▶ spaces: tensors with symmetries
- ▶ operators: beyond grad, curl, div

Benefit

- ▶ automatic well-posedness
- ▶ structure-preserving discretization
- ▶ solver-friendly formulations

Applications

- ▶ elasticity, Cosserat / micropolar models
- ▶ defects (dislocations, disclinations)
- ▶ contact and mixed-dimensional models

COMPLEXES FROM COMPLEXES: CONTINUUM MODELS

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EXAMPLES OF PDES FROM (LINEARIZED) CONTINUUM MECHANICS

Elasticity: deformation and mechanics of solids

elasticity equation:

$$-\operatorname{div}(A \operatorname{def} u) = f.$$

u

$$e := \operatorname{def} u := 1/2(\nabla u + \nabla u^T)$$

$$\sigma := A \operatorname{def} u$$

displacement (vector),
strain (linearized deformation),
stress.

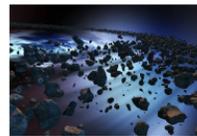
analogy to Poisson equation:

$$-\operatorname{div}(A \operatorname{grad} v) = g.$$

GRANULAR MATERIAL



Granular structures appear across scales: ice floes (grains = icebergs), asteroid belts (grains = asteroids)



Size effects

Classical elasticity: geometrically similar structures behave the same

Real materials: this fails \Rightarrow **need microstructure**

Cosserat models

- ▶ incorporate grain size and rotation
- ▶ lead to geometric concepts (e.g. torsion)

Cartan's bridge between Einstein and the Cosserats (Scholz, 2019)

COMPLEXES FROM COMPLEXES: CONTINUUM MODELS

Energy in the linear model: u : displacement (vector), ω : rotation (axial vector)

$$\begin{aligned}\mathcal{E}^{\text{Cosserat}}(u, \omega) &:= \int_{\Omega} \left(\frac{1}{2} \|\text{grad } u - \text{mskw } \omega\|_{\mathcal{C}_1}^2 + \frac{1}{2} \|\text{grad } \omega\|_{\mathcal{C}_2}^2 - \langle f_u, u \rangle - \langle f_\omega, \omega \rangle \right) dx \\ &= \int_{\Omega} \left(\frac{1}{2} \|\text{sym grad } u\|_{\mathcal{C}}^2 + \mu_c \|1/2 \text{curl } u - \omega\|^2 + \frac{\gamma + \beta}{2} \|\text{sym grad } \omega\|^2 \right. \\ &\quad \left. + \frac{\gamma - \beta}{4} \|\text{curl } \omega\|^2 + \frac{\alpha}{2} \|\text{div } \omega\|^2 \right) dx - \int_{\Omega} \langle f_u, u \rangle + \langle f_\omega, \omega \rangle dx,\end{aligned}$$

$$C_1(\varepsilon) = 2\mu \text{sym } \varepsilon + \lambda \text{tr } \varepsilon I + \mu_c \text{skw } \varepsilon = \mathcal{C}(\varepsilon) + \mu_c \text{skw } \varepsilon, \quad \mathcal{C}(\varepsilon) = 2\mu \text{sym } \varepsilon + \lambda \text{tr } \varepsilon I,$$

$$\begin{aligned}C_2(\varepsilon) &= (\gamma + \beta) \text{sym } \varepsilon + \alpha \text{tr } \varepsilon I + (\gamma - \beta) \text{skw } \varepsilon \\ &= (\gamma + \beta) \text{dev sym } \varepsilon + \frac{3\alpha + \beta + \gamma}{3} \text{tr } \varepsilon I + (\gamma - \beta) \text{skw } \varepsilon,\end{aligned}$$

Structure

- ▶ classical elasticity tensor \mathcal{C} + additional micropolar moduli $(\mu_c, \alpha, \beta, \gamma)$
- ▶ coupling: $\text{grad } u - \text{mskw } \omega$

Geometric origin

$$\text{grad } u - \text{mskw } \omega \quad \Leftarrow \quad \text{linearization of } \exp(\text{mskw } \omega) \in \text{SO}(3)$$

Example of Eringen's **micropolar media**

Open problem: parameter-robust numerical methods

WEAK AND STRONG COUPLING IN COSSERAT

Energy

$$\mathcal{E}^{\text{Cosserat}}(u, \omega) = \int_{\Omega} \left(\frac{1}{2} \|\text{sym grad } u\|_c^2 + \mu_c \|1/2 \text{ curl } u - \omega\|^2 + \frac{1}{2} \|\text{grad } \omega\|_{C_2}^2 \right) dx$$

Weak coupling ($\mu_c = 0$)

- ▶ u and ω decouple
- ▶ classical elasticity for u

Strong coupling ($\mu_c \rightarrow \infty$)

- ▶ $\omega = \frac{1}{2} \text{ curl } u$
- ▶ energy $\sim \|\text{grad curl } u\|^2$
- ▶ couple stress model

Parameter-robust methods must capture both regimes

DEFECTS: ELASTICITY-ELECTROMAGNETISM ANALOGUE

KRÖNER [13] has developed a most useful analogy between the theory of internal stresses and strains as described in sections 2 to 6 and the theory of the magnetic field of distributions of stationary electric currents. Table 1 contains a list of the corresponding physical quantities, differential operators, and equations. We hope that this table is understandable without any further comments (see also the review article by DE WIT [10]).

Table 1
Correspondences in elasticity and magnetism

Elasticity	Magnetism
vector quantity	scalar quantity
tensor rank two	vector
tensor rank four	tensor rank two
Div	div
Ink	curl
Div Ink $\equiv 0$	div curl $\equiv 0$
Def	grad
Ink Def $\equiv 0$	curl grad $\equiv 0$
Burgers vector \mathbf{b}	current I
incompatibility tensor $\boldsymbol{\eta}$	current density \mathbf{J}
strain tensor $\boldsymbol{\epsilon}$	magnetic intensity \mathbf{H}
stress tensor $\boldsymbol{\sigma}$	magnetic induction \mathbf{B}
stress function tensor $\boldsymbol{\chi}, \boldsymbol{\chi}'$	vector potential \mathbf{A}
elastic constants C (or G, K)	permeability μ
displacement \mathbf{s}	scalar potential ψ
equation (3)	$\mathbf{H} = \text{grad } \psi$
equation (5)	curl $\mathbf{H} = \mathbf{J}$
equation (17)	div $\mathbf{B} = 0$
equation (18)	$\mathbf{B} = \text{curl } \mathbf{A}$
equations (19), (19a)	$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$
equation (20)	div $\mathbf{A} = 0$
equation (22)	$\mathbf{A} = \frac{\mu}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')}{ \mathbf{r} - \mathbf{r}' } d\tau_{r'}$

A UNIFYING PERSPECTIVE

All the above models can be written as

Hodge–Laplacian of differential complexes

Examples

- ▶ de Rham complex \Rightarrow electromagnetism, diffusion
- ▶ elasticity (BGG) complexes \Rightarrow linear elasticity
- ▶ twisted complexes \Rightarrow Cosserat / micropolar models
- ▶ high-order forms \Rightarrow defects, higher-order models

DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

$$\text{RM} \xrightarrow{\mathbb{C}} \mathcal{C}^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{inc}} \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{div}} \mathcal{C}^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

embedding $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

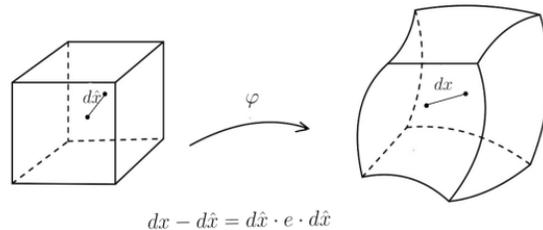
change of metric (strain)

$$\text{RM} \xrightarrow{\subset} C^\infty \otimes \mathbb{R}^3 \xrightarrow{\text{sym grad}} C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{R}^3 \longrightarrow 0$$

$$\varphi \longrightarrow e = (\widehat{\nabla} \varphi) \cdot (\varphi \widehat{\nabla}) - I$$

$e = 0$ iff φ is a rigid body motion.

Linearisation: $e = \text{sym grad } u$, in terms of displacement $u(\hat{x}) = \varphi(\hat{x}) - \hat{x}$.



DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

metric (strain)

Riemann curvature

$$\begin{array}{ccccccc}
 \text{RM} & \xrightarrow{\subset} & \mathcal{C}^\infty \otimes \mathbb{R}^3 & \xrightarrow{\text{sym grad}} & \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} & \xrightarrow{\text{inc}} & \mathcal{C}^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} & \xrightarrow{\text{div}} & \mathcal{C}^\infty \otimes \mathbb{R}^3 & \longrightarrow & 0 \\
 & & & & e & \longrightarrow & \text{Riem}(e) & & & &
 \end{array}$$

Strain tensor (change of metric) $e = (\widehat{\nabla} \boldsymbol{\varphi}) \cdot (\boldsymbol{\varphi} \widehat{\nabla}) - I$ satisfies $\text{Riem}(e) = 0$.

Defect theory: Kröner et al. used violation of compatibility conditions to model defects and incompatibility

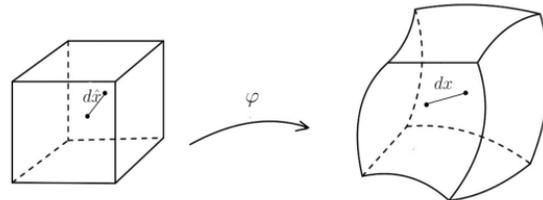
Linearisation: Saint-Venant compatibility condition $\text{inc } e := \nabla \times e \times \nabla = 0$.



Bernhard Riemann



Ekkehart Kröner



$$dx - d\hat{x} = d\hat{x} \cdot e \cdot d\hat{x}$$

DIFFERENTIAL STRUCTURES IN ELASTICITY

Linear elasticity (Calabi, Kröner) complex

curvature / stress

covector / force

$$\begin{array}{ccccccc}
 \text{RM} & \xrightarrow{\mathcal{C}} & C^\infty \otimes \mathbb{R}^3 & \xrightarrow{\text{sym grad}} & C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} & \xrightarrow{\text{inc}} & C^\infty \otimes \mathbb{R}_{\text{sym}}^{3 \times 3} & \xrightarrow{\text{div}} & C^\infty \otimes \mathbb{R}^3 & \longrightarrow & 0 \\
 & & & & & & \sigma & \longrightarrow & \nabla \cdot \sigma & &
 \end{array}$$

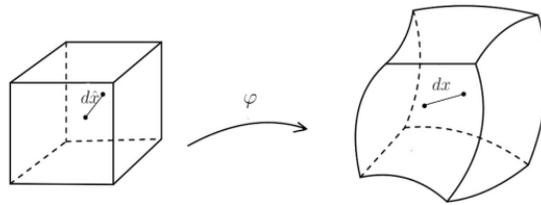
Cauchy stress tensor σ balances load $\text{div } \sigma = f$ with $\sigma = Ae$ (Hooke's law); incompatibility causes internal stress $\text{inc } e$.



Robert Hooke



Augustin-Louis Cauchy



COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

► Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 1: connect two (or more) de Rham complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0 \\ & & & & \nearrow S^0 & & \nearrow S^1 & & \nearrow S^2 & & \\ 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0 \end{array}$$

S^\bullet : algebraic operators, connecting components of vectors/matrices

COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

► Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 2: elimination

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{S} + \mathbb{K} & \xrightarrow{\text{curl}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0 \\
 & & & \nearrow -\text{mskw} & & \nearrow \mathbb{S} & & \nearrow 2 \text{vskw} & & & \\
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{R}^{3 \times 3} & \xrightarrow{\text{curl}} & \mathbb{S} + \mathbb{K} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0
 \end{array}$$

\mathbb{S} : symmetric matrix, \mathbb{K} : skew-symmetric matrix

COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: [complexes from complexes](#)

► Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 2: elimination

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathbb{S} + \mathbb{K} & \xrightarrow{\text{curl}} & \cancel{\mathbb{R}^{3 \times 3}} & \xrightarrow{\text{div}} & \cancel{\mathbb{R}^3} & \longrightarrow & 0 \\
 & & & \swarrow -\text{mskw} & & \nearrow \mathbb{S} & & \swarrow 2\text{vskw} & & & \\
 0 & \longrightarrow & \cancel{\mathbb{R}^3} & \xrightarrow{\text{grad}} & \cancel{\mathbb{R}^{3 \times 3}} & \xrightarrow{\text{curl}} & \mathbb{S} + \mathbb{K} & \xrightarrow{\text{div}} & \mathbb{R}^3 & \longrightarrow & 0
 \end{array}$$

\mathbb{S} : symmetric matrix, \mathbb{K} : skew-symmetric matrix

COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

- ▶ Douglas Arnold, KH, *Complexes from complexes*, Foundations of Computational Mathematics (2021)

Step 3: connect rows by zig-zag

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{sym grad}} & \mathbb{S} & \xrightarrow{\text{curl}} & \\ & & & & & \swarrow & \\ & & & & & \text{curl}^T & \\ & & & & & \longleftarrow & \\ & & & & \mathbb{S} & \xrightarrow{\text{div}} & \mathbb{R}^3 \longrightarrow 0. \end{array}$$

Conclusion: cohomology of the output (elasticity) is isomorphic to the input (de Rham)

Analytic results follow: Poincaré–Korn inequalities, Hodge decomposition, compactness...

fitting Sobolev spaces in diagrams; operators have closed range (= kernel + harmonics); $d-d^*$ lemma

Inspired by the Bernstein-Gelfand-Gelfand (BGG) construction (B-G-G 1975, Čap, Slovák, Souček 2001, Eastwood 2000, Arnold, Falk, Winther 2006, Arnold, KH 2021, Čap, KH 2023)

COMPLEXES FROM COMPLEXES

Generating, analysing and discretising linear (deformation) complexes: *complexes from complexes*

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Step 3: connect rows by zig-zag

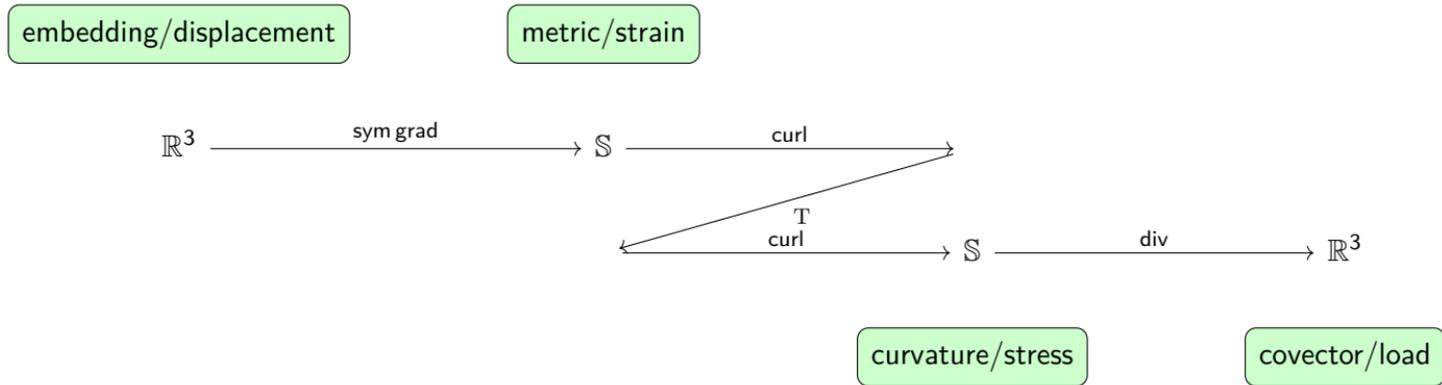
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{R}^3 & \xrightarrow{\text{sym grad}} & \mathbb{S} & \xrightarrow{\text{curl}} & \\ & & & & & \swarrow & \\ & & & & & \text{curl}^T & \\ & & & & \mathbb{S} & \xrightarrow{\text{div}} & \mathbb{R}^3 \longrightarrow 0. \end{array}$$

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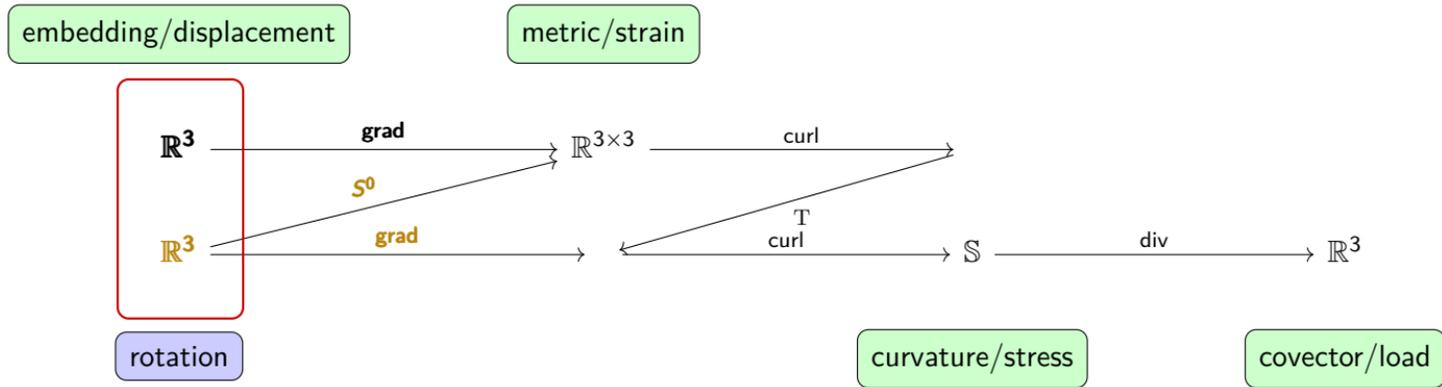
But, is it purely mathematical?

CALABI COMPLEX

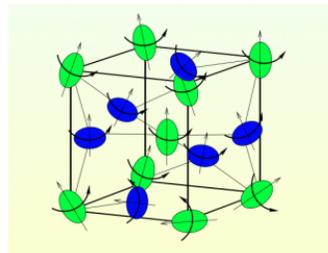


Riemann, Kröner, Cauchy, Hooke

CALABI COMPLEX



Cosserat brothers

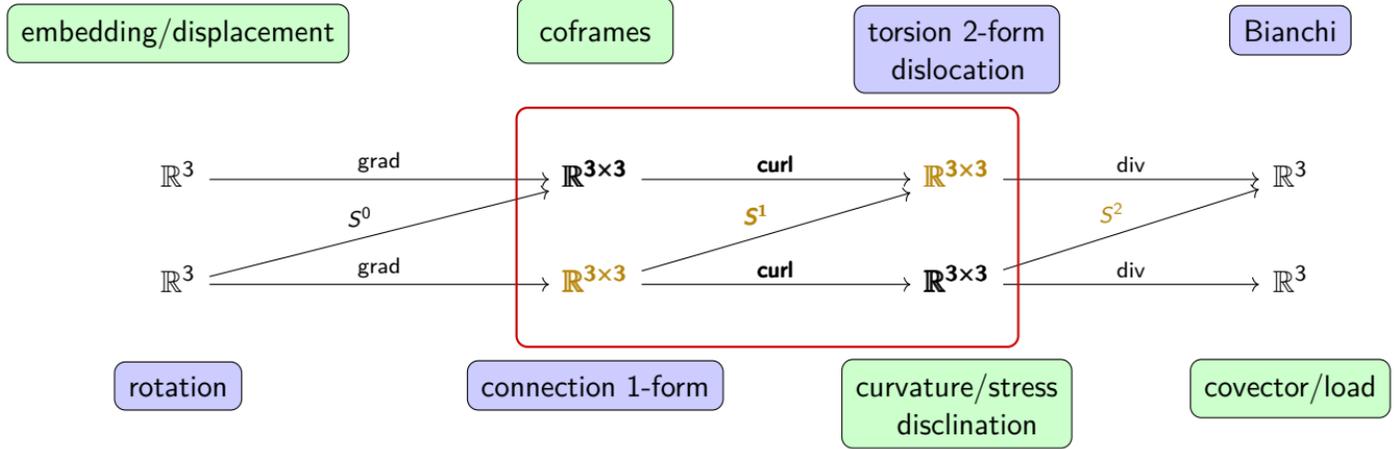


Cosserat continua: microstructures (rotation, stretch etc.)

Observations: A. Čap & KH, *BGG sequences with weak regularity and applications*. FoCM (2024).

Leading to **first parameter-robust scheme for Cosserat model**: A.Dziubek, KH, M.Karow & M. Neunteufel, arXiv (2024).

CALABI COMPLEX



Élie Cartan

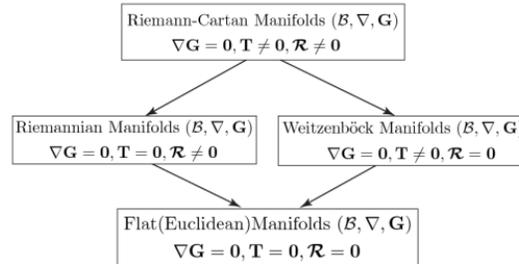


Arash Yavari



Alain Goriely

Cartan's bridge between Einstein and Cosserat brothers – torsion



Riemann-Cartan Geometry of Nonlinear Dislocation Mechanics,
Yavari and Goriely, ARMA (2012)

Observations: Christiansen, KH, & Lin, *Extended Regge complex for linearized Riemann-Cartan geometry and cohomology*. arXiv (2023). **BGG construction is thus cohomology-preserving elimination of microstructures!**

HOW TO DERIVE MORE COMPLEXES: THE BGG MACHINERY

Bernstein-Gelfand-Gelfand (BGG) machinery: Derive complexes from de Rham complexes; carry over de Rham results. (B-G-G 1975, Čap,Slovák,Souček 2001, Eastwood 2000, Arnold,Falk,Winther 2006, Arnold,Hu 2021, Čap,Hu 2023)

BGG diagram: complexes connected by algebraic operators in a (anti)commuting diagram ($dS = -Sd$)

$$\begin{array}{ccccccccccc}
 \dots & \longrightarrow & V^{k-2} & \xrightarrow{d^{k-2}} & V^{k-1} & \xrightarrow{d^{k-1}} & V^k & \xrightarrow{d^k} & V^{k+1} & \longrightarrow & \dots \\
 & & & \nearrow S^{k-2} & & \nearrow S^{k-1} & & \nearrow S^k & & & \\
 \dots & \longrightarrow & W^{k-2} & \xrightarrow{d^{k-2}} & W^{k-1} & \xrightarrow{d^{k-1}} & W^k & \xrightarrow{\text{div}} & W^{k+1} & \longrightarrow & \dots
 \end{array}$$

Two complexes can be derived from the above BGG diagram:

twisted complex:

$$\dots \longrightarrow \begin{pmatrix} V^{k-1} \\ W^{k-1} \end{pmatrix} \xrightarrow{\begin{pmatrix} d^{k-1} & -S^{k-1} \\ 0 & d^{k-1} \end{pmatrix}} \begin{pmatrix} V^k \\ W^k \end{pmatrix} \xrightarrow{\begin{pmatrix} d^k & -S^k \\ 0 & d^k \end{pmatrix}} \begin{pmatrix} V^{k+1} \\ W^{k+1} \end{pmatrix} \longrightarrow \dots$$

BGG diagram: eliminating components connected by S^\bullet

COMPLEXES V.S. MECHANICS MODELS

BGG diagram in 1D:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^2 & \xrightarrow{\partial_x} & H^1 & \longrightarrow & 0 \\
 & & & & \nearrow I & & \\
 0 & \longrightarrow & H^1 & \xrightarrow{\partial_x} & L^2 & \longrightarrow & 0.
 \end{array}$$

Twisted complex:

$$0 \longrightarrow \begin{pmatrix} H^1 \\ H^1 \end{pmatrix} \xrightarrow{\begin{pmatrix} \frac{d}{dx} & -I \\ 0 & \frac{d}{dx} \end{pmatrix}} \begin{pmatrix} L^2 \\ L^2 \end{pmatrix} \longrightarrow 0.$$

Energy of Hodge-Laplacian:

$$\mu \left\| \frac{d}{dx} w - \varphi \right\|_{C_1}^2 + \left\| \frac{d}{dx} \varphi \right\|_{C_2}^2$$

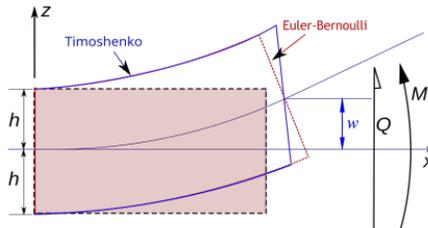
BGG complex:

$$0 \longrightarrow H^2 \xrightarrow{\partial_x^2} L^2 \longrightarrow 0.$$

Energy of Hodge-Laplacian

$$\left\| \frac{d^2}{dx^2} w \right\|_{C}^2.$$

Eliminating φ or $\mu \rightarrow \infty$

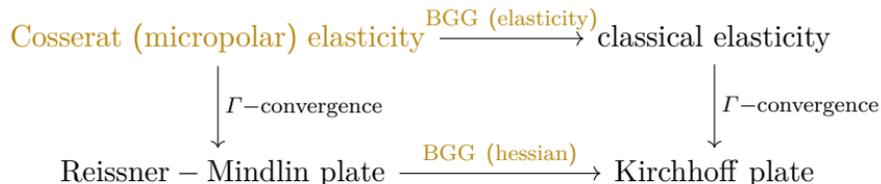


TWISTED VS. BGG COMPLEXES: MICROSTRUCTURE AND REDUCTION

Hodge Laplacian:

	twisted complex	BGG complex
1D	Timoshenko beam	Euler-Bernoulli beam
2D	Reissner-Mindlin plate	Kirchhoff-Love plate
3D	Cosserat elasticity	classical elasticity

Reminder: **twisted complex** has all spaces in a two-row diagram; **BGG complex** is after the elimination



Γ convergence: *The Reissner-Mindlin plate is the Γ -limit of Cosserat elasticity.* Neff, Hong, & Jeong M3AS, (2010).

Mechanics interpretation of BGG construction: eliminating microstructure variables (e.g., pointwise rotation) or torsion from twisted complexes via cohomology-preserving projections.

DOUBLE COMPLEXES: MIXED DIMENSIONAL GEOMETRY

1	De Rham complexes: electromagnetism	1
2	Complexes from complexes: continuum models	11
3	Double complexes: mixed dimensional geometry	23

MIXED/MULTI-DIMENSIONAL MODELS: OPERATOR δ FOR JUMP AT INTERFACE

Let $\{U_i\}_i$ be an open cover of Ω , with

$$U_{i_0 \dots i_s} := U_{i_0} \cap \dots \cap U_{i_s}.$$

Define

$$\mathcal{W}^{k,s} := \bigoplus_{i_0, \dots, i_s} \Lambda^k(U_{i_0 \dots i_s}),$$

the space of k -forms on all $(s+1)$ -fold intersections.

There are two natural operators:

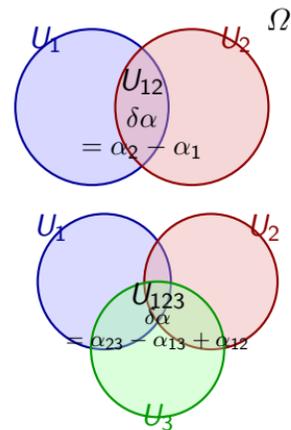
$$d : \mathcal{W}^{k,s} \rightarrow \mathcal{W}^{k+1,s}, \quad \delta : \mathcal{W}^{k,s} \rightarrow \mathcal{W}^{k,s+1},$$

with

$$(\delta\alpha)_{i_0, \dots, i_{s+1}} = \sum_{j=0}^{s+1} (-1)^j \alpha_{i_0, \dots, \widehat{i_j}, \dots, i_{s+1}} \Big|_{U_{i_0 \dots i_{s+1}}}.$$

Example: two subdomains. If $\Omega = U_1 \cup U_2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathcal{W}^{k,0}$, then

$$(\delta\alpha)_{1,2} = (\alpha_2 - \alpha_1) \Big|_{U_{1,2}}.$$



THE ČECH-DE RHAM COMPLEX

The exterior derivative and Čech differential satisfy

$$d^2 = 0, \quad \delta^2 = 0, \quad d\delta = \delta d.$$

Hence we obtain a double complex:

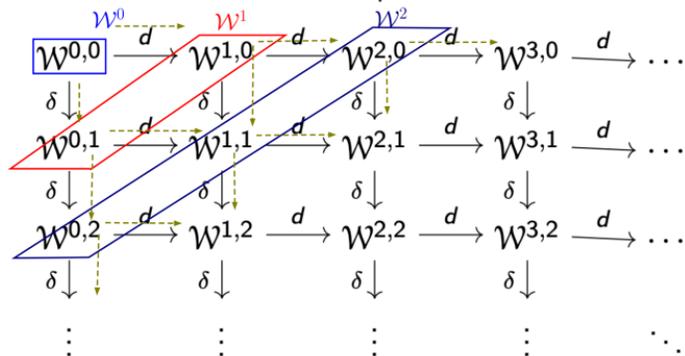
$$\begin{array}{ccccccc} \mathcal{W}^{0,0} & \xrightarrow{d} & \mathcal{W}^{1,0} & \xrightarrow{d} & \mathcal{W}^{2,0} & \xrightarrow{d} & \mathcal{W}^{3,0} \xrightarrow{d} \dots \\ \delta \downarrow & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ \mathcal{W}^{0,1} & \xrightarrow{d} & \mathcal{W}^{1,1} & \xrightarrow{d} & \mathcal{W}^{2,1} & \xrightarrow{d} & \mathcal{W}^{3,1} \xrightarrow{d} \dots \\ \delta \downarrow & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ \mathcal{W}^{0,2} & \xrightarrow{d} & \mathcal{W}^{1,2} & \xrightarrow{d} & \mathcal{W}^{2,2} & \xrightarrow{d} & \mathcal{W}^{3,2} \xrightarrow{d} \dots \\ \delta \downarrow & & \delta \downarrow & & \delta \downarrow & & \delta \downarrow \\ \vdots & & \vdots & & \vdots & & \vdots \quad \ddots \end{array}$$

THE ČECH–DE RHAM COMPLEX

The exterior derivative and Čech differential satisfy

$$d^2 = 0, \quad \delta^2 = 0, \quad d\delta = \delta d.$$

Hence we obtain a double complex:



Define the total space

$$\mathcal{W}^k := \bigoplus_{p+q=k} \mathcal{W}^{p,q},$$

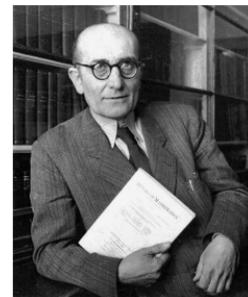
and the total differential

$$D = d + (-1)^p \delta \quad \text{on } \mathcal{W}^{p,q}.$$

Then $D^2 = 0$, so

$$\dots \longrightarrow \mathcal{W}^{k-1} \xrightarrow{D^{k-1}} \mathcal{W}^k \xrightarrow{D^k} \mathcal{W}^{k+1} \longrightarrow \dots$$

is the Čech–de Rham complex.



Eduard Čech

FROM ČECH COMPLEX TO COUPLED PDES

Idea: the Čech complex induces a coupled energy.

Example: $\mathcal{U} = \{U_1, U_2\}$

$$\begin{array}{c} \mathcal{W}^{0,0} \xrightarrow{d} \mathcal{W}^{1,0} \\ \downarrow \delta \\ \mathcal{W}^{0,1} \end{array}$$

For $u \in \mathcal{W}^{0,0}$:

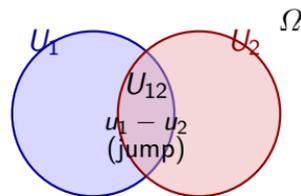
$$E(u) = \|\nabla u\|_{U_1}^2 + \|\nabla u\|_{U_2}^2 + \|u_1 - u_2\|_{U_{12}}^2 + (f, u).$$

Euler–Lagrange equations:

$$-\Delta u_i + \chi_{U_{12}}(u_i - u_j) = f \quad \text{in } U_i, \quad i \neq j.$$

Interpretation:

$d \Rightarrow$ diffusion, $\delta \Rightarrow$ penalty / coupling.



W. M. Boon, D. F. Holmen, J. M. Nordbotten, J. E. Vatne, *The Hodge-Laplacian on the Čech–de Rham complex governs coupled problems*, J. Math. Anal. Appl., 2025.

W. M. Boon, J. M. Nordbotten, *Mixed-dimensional poromechanical models of fractured porous media*, Acta Mechanica (2023).

THREE TYPES OF COUPLING MECHANISMS

(1) Contact (mechanics)

$$\delta u = u_1 - u_2 \quad \text{on interface}$$

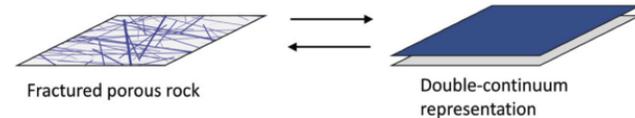
- ▶ jump contributes to strain
- ▶ models contact / interface mechanics



(2) Full overlap (multi-media)

$$\delta u = u_1 - u_2 \quad \text{in } \Omega$$

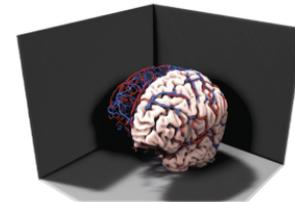
- ▶ two fields on the same domain
- ▶ e.g. double-continuum porous media



(3) Mixed-dimensional coupling

$$\delta u = u_\Gamma - u_\Omega$$

- ▶ coupling across dimensions
- ▶ fracture / network



Common structure: local physics + $\delta(\text{interaction})$

W. M. Boon, D. F. Holmen, J. M. Nordbotten, J. E. Vatne, *The Hodge-Laplacian on the Čech-de Rham complex governs coupled problems*, J. Math. Anal. Appl., 2025.

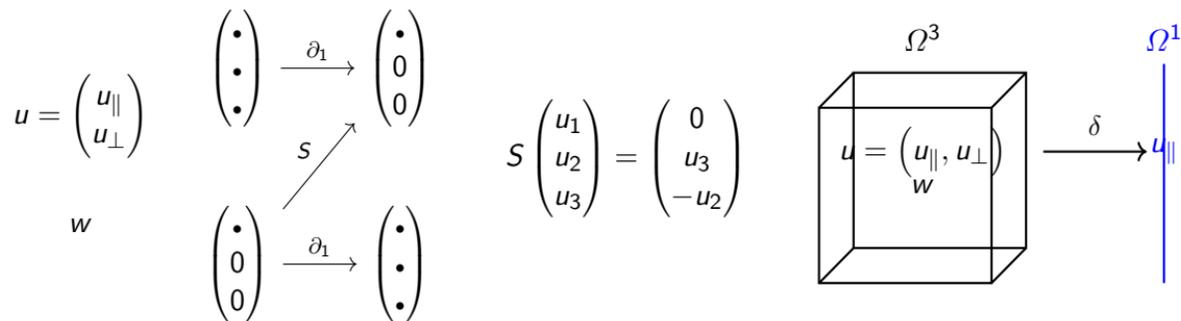
W. M. Boon, J. M. Nordbotten, *Mixed-dimensional poromechanical models of fractured porous media*, Acta Mechanica (2023).

ONGOING WORK: ČECH + BERNSTEIN–GELFAND–GELFAND

Goal: Čech + BGG \Rightarrow mixed-dimensional coupling + elasticity / microstructure

Example: 3D–1D coupling

BGG reduction: \mathbb{R}^3 -valued \mathbb{R} -forms; algebraic+analytic framework in Čap-Hu 2021



Reduced operator: Čap-Hu 2021

$$\mathcal{D} : \begin{pmatrix} u \in C^\infty \otimes \mathbb{R}^3 \\ w \in C^\infty \end{pmatrix} \mapsto \begin{pmatrix} (\partial_{\parallel} u_{\parallel}, 0) & \text{1D elasticity} \\ \begin{pmatrix} \partial_{\parallel} w \\ -\partial_{\parallel}^2 u_2^\perp \\ \partial_{\parallel}^2 u_1^\perp \end{pmatrix} & \begin{matrix} \text{rotation} \\ \text{bending} \\ \text{bending} \end{matrix} \end{pmatrix}.$$

Then add δ operators + Čech structures... Subtleties in Sobolev setting.

OPEN DIRECTIONS

Nonlinear geometry

exactness: rigidity
two motions induce same metric iff up to RM

rigid body motion $\xrightarrow{\subset}$ map \mathbb{R}^3 to \mathbb{R}^3 $\xrightarrow{\varphi \mapsto \varphi^* g_0 - g_0}$ metric $\xrightarrow{\text{Ricci}}$ curvature

exactness: fundamental thm of Riem geometry
metric has vanishing curvature iff metric is Euclidean

Curved setting

Thermodynamics: mechanics = **geometry** + physics (constitutive law, Hodge star)

SUMMARY

Differential complexes provide a unified geometric framework for

geometry / deformation	BGG machinery
microstructure	twisted complexes
coupling, multi/mixed dimensions	Čech complex
defects	higher-order forms

Outlook

- ▶ interaction of Čech and BGG structures
- ▶ nonlinear and curved (Riemannian) geometry
- ▶ discretization and solvers

References

- ▶ *Complexes from complexes*, D.N. Arnold, KH; *FoCM* (2021)
framework, analytic results from homological structures
- ▶ *BGG sequences with weak regularity and applications*, A. Čap, KH; *FoCM* (2023)
general framework, conformal complexes
- ▶ *Nonlinear elasticity complex and a finite element diagram chase*, KH; *Springer INdAM* (2023)
nonlinear geometry, diagram chase
- ▶ *Extended Regge complex for linearized Riemann-Cartan geometry*, Christiansen, KH, Lin; *FoCM* (2026)
torsion, defects